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Nonlinear Wave Number of an Electron Plasma Wave*

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The wave number of an electron plasma wave propagating on a collisionless plasma column is found to differ from that of a small-amplitude wave at the same frequency. The nonlinear wave-number shift oscillates in space. The magnitude of its first maximum, when normalized to the initial (nonlinear) damping coefficient, is found to be proportional to the square root of the initial wave amplitude.

Recent theories[1-3] predict that the wave number, $k_r = 2\pi/\lambda$, of a large-amplitude electron plasma wave is not the same as the wave number of a small-amplitude wave at the same frequency. We have measured[4] the nonlinear wave-number shift using a phase-tracking interferometer. The spatial dependence, magnitude, and scaling with frequency and amplitude are determined.

If the initial wave amplitude, Φ (electric potential), is small enough, the linear theory of Landau[5] applies and is well verified by experiment[6,7]; in a homogeneous plasma a wave of constant frequency has a complex wave number $k = k_r + ik_i$, which is independent of position and Φ. It is also well established[8] in theory and experiment that k_i becomes a function of position and Φ when the wave amplitude is large enough to trap the resonant electrons. Conservation of energy between the wave and the trapped electrons leads to nonlinear amplitude oscillations. The trapped electrons also cause k_r to depend on position and Φ. Conservation of energy in a reference frame moving with the linear-wave phase velocity, v_p, leads to nonlinear oscillations in k_r. In this frame, the amplitude oscillations are related to conservation of momentum.

Morales and O'Neil[1] considered the initial-value problem. A wave of fixed wavelength is turned on at $t = 0$ and propagates in an infinite, homogeneous, one-dimensional, collisionless plasma. The time-dependent shift in the complex frequency caused by the trapped electrons is calculated. They assume that the wave amplitude is constant (negligible linear damping) and small enough for the resonant electrons to be represented by a second-order Taylor expansion of the electron velocity distribution function, $f_0(v)$, about v_p. They also give the transformation to the boundary-value problem studied in our experiment. In this case a wave of fixed frequency ω, determined by a single generator, propagates away from a transmitter probe at $z = 0$. The time-dependent frequency shift becomes a space-dependent shift in the complex wave number. The imaginary part of the shift gives the amplitude oscillations. The real part is

$$\delta k_r(z) = -(\Omega_0/v_g)g(k_B z), \tag{1}$$

where $\Omega_0 = (e\Phi/m)^{1/2}(\omega_p/k_r)^2(\partial^2 f_0/\partial v^2)_{v_p}/(\partial\epsilon/\partial\omega)_{\omega,k_r}$, ϵ is the plasma dielectric function, k_r is the linear wave number, ω is the (constant) oscillation frequency, v_g is the linear group velocity, ω_p is the plasma frequency, and $-e$ and m are the charge and mass of the electron. The function $g(k_B z)$ (Fig. 2 of Ref. 1) gives the space dependence in terms of the scaled position $k_B z$, where $k_B = \omega_B/v_p$ and $\omega_B = (ek_r^2\Phi/m)^{1/2}$ is the bounce frequency of the trapped electrons. If $f_0(v)$ is Maxwellian, $\Omega_0 > 0$. Since $g(k_B z) \leq 0$, Eq. (1) gives $\delta k_r \geq 0$. Examination of $g(k_B z)$ shows that δk_r is initially zero; it increases and executes damped oscillations about an asymptotic value. The oscillation rate is twice that of the nonlinear amplitude oscillations. The position, L, of the first maximum of δk_r (given by $k_B L = \text{const}$) is near the position of the first amplitude minimum.

Equation (1) is not well suited for comparison with experiment. The major frequency dependence appears through $(\partial^2 f_0/\partial v^2)_{v_p}$ which is difficult to measure accurately. Using the spatial Landau damping coefficient,[9] $k_i = -(\pi/v_g)(\omega_p/k_r)^2 \times(\partial f_0/\partial v)_{v_p}/(\partial\epsilon/\partial\omega)_{\omega,k_r}$, Eq. (1) can be written in a more suitable form,

$$\delta k_r(z) = (e\Phi/m)^{1/2} k_i R g(k_B z)/\pi, \qquad (2)$$

where $R = (\partial^2 f_0/\partial v^2)_{v_p}/(\partial f_0/\partial v)_{v_p}$. The major frequency dependence now appears through k_i which can be accurately measured. If $f_0(v)$ is nearly Maxwellian, the frequency dependence of R is slow compared to the exponential dependence of k_i on v_p^2.

The assumptions of the theory are not completely satisfied in our experiment. At small amplitudes the waves studied have finite linear damping, i.e., the assumption of negligible linear damping is not satisfied. At larger amplitudes the measured initial nonlinear damping coefficient, $k_i(\Phi)$, increases with wave amplitude[10]; Φ is then too large for the second-order Taylor expansion of $f_0(v)$ to be valid. In addition, the wave amplitude and electron density depend on radius in the cylindrical geometry of the experiment. We have estimated this correction and find that the scaling of Eq. (2) is not changed. The spatial dependence is modified; however, the qualitative behavior noted above remains the same. Surprisingly, only a slight modification of Eq. (2) is necessary to describe our data.

Other theoretical work finds an increase in the magnitude of δk_r above that given by Eq. (2). Lee and Pocobelli[2] include small linear damping and find an increase of up to 50%. Tsai,[3] by simulating the resonant electrons, is able to include linear damping, $k_i(\Phi)$, and the complex wave-number shift. He finds an increase of up to a factor of 2. Comparison with experiment is difficult because an analytic expression for the scaling of δk_r is not available.

The experimental apparatus, described in detail elsewhere,[11] produces a collisionless hydrogen plasma column surrounded by a conducting cylinder (5.2 cm radius) and immersed in a uniform axial magnetic field (200 G). Probes can be inserted into the plasma and moved radially and axially. The plasma temperature ($T = 8.2$ eV) and central density (3.0×10^8 cm^{-3}) are known (to 5%) from the measured dispersion of linear waves.[6] The measured Landau damping of linear waves shows that $f_0(v)$ is Maxwellian in the range of the wave phase velocities. The damping coefficients, k_i and $k_i(\Phi)$, are obtained (to 10%) from the measured dependence of wave power on position (power bandwidth 3 MHz, $\Delta\omega/\omega < 3\%$—this measurement is dominated by the power in the launched plasma wave at frequency ω). For comparison with the theory, Φ is taken to be the initial wave amplitude at $r=0$. It is obtained from the calculated wave energy density and the power-coupling coefficient of the transmitter probe measured by the three-probe coupling technique.[12] The uncertainty in the absolute magnitude of $\sqrt{\Phi}$ is 20%. At a single frequency the relative magnitude of $\sqrt{\Phi}$ is known to 5%.

The spatial resolution of the conventional interferometer technique for measuring wavelength is not adequate for this experiment. We used a phase-tracking interferometer[13] to measure the wave phase accurately. It is a homodyne interferometer with a feedback-controlled rf resolver (phase shifter) in the transmitter circuit. Homodyning is required by the narrow bandwidth of the resolver. As the receiver probe moves, the feedback control loop continuously adjusts the resolver so that the output of the interferometer is zero. The phase shift inserted by the resolver is then equal to the wave phase plus an unknown constant phase due to rf circuit elements. The phase shift is accurately and continuously measured as a function of position.

To see the nonlinear part of the phase directly, the linear phase, $k_r z$, of a wave of constant wave number is electronically subtracted from the measured phase. The resulting relative phase, $\Delta\theta(z)$, is recorded on an X-Y recorder as a function of position. The position and $\Delta\theta(z)$ are also converted to digital form and stored on magnetic tape.

To determine $\delta k_r(z)$ a subtraction procedure is used to remove the wave-number variations caused by small inhomogeneities in the plasma column. A digital computer is used to subtract $\Delta\theta(z)$ of a linear wave from $\Delta\theta(z)$ of a nonlinear wave at the same frequency. The result is averaged and differentiated to give $\delta k_r(z)$ with an accuracy of ± 0.01 cm^{-1}.

Figure 1 shows some typical raw data. Since the measured phase contains an unknown constant phase, the curves have been moved vertically so that they begin with the same phase near the transmitter. The small bump at the beginning of

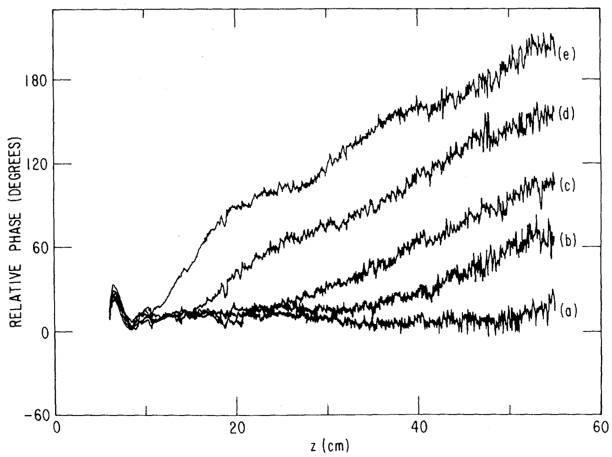

FIG. 1. The relative phase as a function of position. The transmitter is at $z=0$. $\omega/2\pi = 135$ MHz, $k_r = 1.83$ cm^{-1}, $e\Phi/T$ is (a) 0.0075, (b) 0.067, (c) 0.16, (d) 0.33, (e) 0.56.

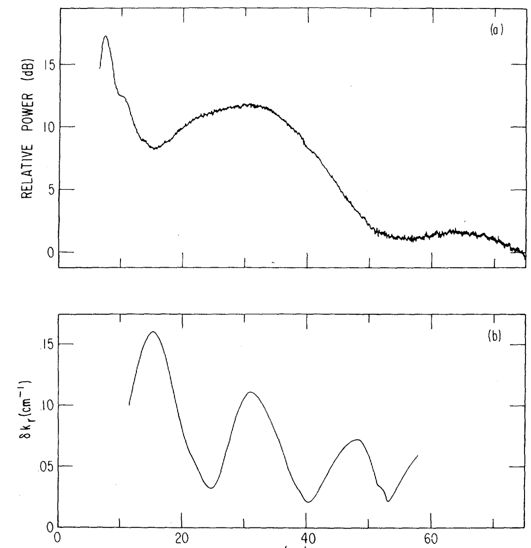

FIG. 2. (a) The wave power and (b) $\delta k_r(z)$ for the data of Fig. 1(e).

each curve is due to direct coupling between the transmitter and receiver probes. The small changes in the slope of $\Delta\theta(z)$ of a linear wave, Fig. 1(a), corresponds to axial density changes of $\pm 1\%$, i.e., the density is axially uniform. Qualitative agreement with the theory can be seen. Near the transmitter the slope of $\Delta\theta(z)$ is independent of Φ and agrees with that of a linear wave, i.e., the initial wave number is the linear wave number. The nonlinear $\Delta\theta(z)$ peels away from the linear $\Delta\theta(z)$ at a position which moves toward the transmitter as Φ increases. This establishes the nonlinear origin of the change in $\Delta\theta(z)$; any amplitude-dependent change in the plasma density would cause $\Delta\theta(z)$ to deviate from the linear phase at $z=0$. Several phase oscillations can be seen in Fig. 1(e).

The space dependence of $\delta k_r(z)$, Fig. 2, is in qualitative agreement with the theory. The oscillation rate is twice that of the amplitude oscillations. The first maximum of δk_r is located near the position of the first amplitude minimum. Data at many frequencies and amplitudes show that the position of the first maximum of δk_r is proportional to $(k_B)^{-1} \propto (\Phi)^{-1/2}$ as expected from the theory.

The first maximum of δk_r is used to determine the scaling with amplitude and frequency. As expected from Eq. (2), $\delta k_r/(k_i R)$ is independent of frequency; however, it increases with amplitude more rapidly than $\sqrt{\Phi}$ and, at the largest amplitudes used, is about a factor of 3 larger than predicted by Eq. (2). The additional amplitude dependence is removed if δk_r is normalized to the initial nonlinear damping coefficient, $k_i(\Phi)$; $\delta k_r/k_i(\Phi)R$ is still independent of frequency and is roughly proportional to $\sqrt{\Phi}$, i.e., $\delta k_r \propto \sqrt{\Phi}k_i(\Phi)R$. Figure 3 shows remarkable agreement between the measured δk_r and the prediction of Eq. (2) when k_i is replaced by $k_i(\Phi)$. This substitution amounts to multiplying Eq. (2) by a factor $S(\Phi)$

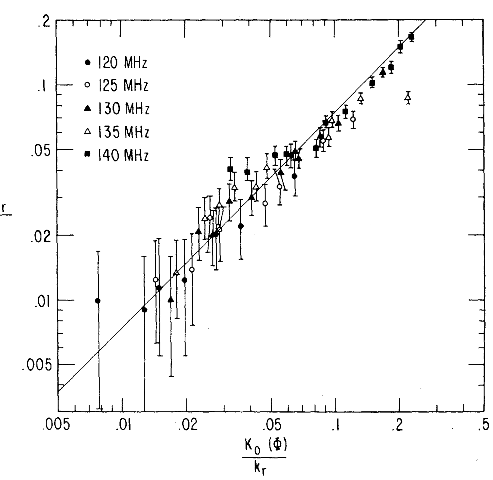

FIG. 3. The fractional change in k_r at the first maximum of $\delta k_r(z)$. $K_0(\Phi) = (e\Phi/m)^{1/2}k_i(\Phi)R$. The wave parameters are, for 120 MHz, $k_r = 1.45$ cm^{-1}; $k_i = 0.0079$ cm^{-1}; for 125 MHz, $k_r = 1.58$ cm^{-1}, $k_i = 0.020$ cm^{-1}; for 130 MHz, $k_r = 1.72$ cm^{-1}, $k_i = 0.32$ cm^{-1}; for 135 MHz, $k_r = 1.83$ cm^{-1}, $k_i = 0.044$ cm^{-1}; for 140 MHz, $k_r = 1.95$ cm^{-1}, $k_i = 0.072$ cm^{-1}.

$=k_i(\Phi)/k_i$ which our data show to be independent of frequency. The line is obtained using $g(k_B z)$ evaluated at the first maximum of δk_r [$k_B z = 1.31\pi$, $g(k_B z) = -2.32$]. The error bars show the ± 0.01-cm^{-1} uncertainty in δk_r. The close agreement suggests that the increase in magnitude found by Refs. 2 and 3 may be related to the amplitude dependence of $k_i(\Phi)$.

In summary, we have measured the nonlinear wave-number shift, δk_r, of a large-amplitude electron plasma wave. The observed spatial oscillations of δk_r agree qualitatively with the theory. The magnitude of the first maximum of δk_r is proportional to $k_i(\Phi)$, the initial nonlinear damping coefficient. This is not predicted by the theory. Even though their assumptions are not satisfied, the magnitude of the first maximum of δk_r is correctly given by the expression of Morales and O'Neil [Eq. (2)] *provided* that the linear damping coefficient is replaced by $k_i(\Phi)$.

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