Viscosity and Thermal Conductivity of Superfluid ^{3}He: Low-Temperature Limit*

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We calculate the viscosity η and diffusive thermal conductivity κ_D of the Balian-Werthamer state of a p-wave superfluid for $T \ll T_c$. The nature of the excitations in the superfluid state is fully taken into account, and realistic effective interactions are used. η tends to a constant, $\sim\frac{1}{4}\eta(T_c)$ for liquid ^{3}He, and $\kappa_D \sim T^{-1}$. Mean free paths are also calculated.

The viscosity in the A and B phases of superfluid ^{3}He has recently been measured by Alvesalo *et al.*[1] and by Johnson *et al.*[2] In this Letter we describe calculations of the normal-fluid viscosity and diffusive thermal conductivity for liquid ^{3}He in the B phase well below the superfluid transition temperature T_c, assuming this phase may be identified with the Balian-Werthamer[3] (BW) state of a p-wave superfluid. There is now considerable evidence for this identification,[4] particularly from recent magnetic resonance measurements.[5,6] Calculations of the transport coefficients have previously been reported,[7,8] but neither of them can be compared quantitatively with the experimental results. Shumeiko's[7] calculations are for an s-wave superfluid and a contact interaction, and Seiden's[8] take into account the true nature of the excitations only approximately.

The basic elementary excitations in a superfluid are linear combinations of particles and holes, and we shall find it convenient here to work in terms of positive-energy excitations only. For describing low-frequency, long-wavelength kinetic phenomena one may use a Boltzmann equation. In addition to processes involving the scattering of two excitations, the collision term contains contributions from others in which three excitations coalesce to produce one, and one excitation decays into three. In general all these processes are equally important, but at temperatures T small compared to Δ/k_B the two-excitation scattering processes are more important by a factor $\exp(\Delta/k_BT)$ than the other processes, which can therefore be neglected. Here Δ is the superfluid gap.

Let us now calculate the relaxation time τ_1 for excitations of momentum $\vec{p}_1$. Consider a situation in which the distribution function differs from its equilibrium value only for excitations in the state 1: τ_1 is then defined by $\partial\delta n_1/\partial t = -\delta n_1/\tau_1$, where δn_1 is the deviation from equilibrium. One finds

$$\tau_1^{-1} = \sum_{2,3,4} W_s(1,2;3,4)[n_2(1-n_3)(1-n_4)+(1-n_2)n_3n_4]\,\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4}\,\delta(E_1+E_2-E_3-E_4). \quad (1)$$

Here $n_i = [\exp(E_i/k_BT)+1]^{-1}$ is the equilibrium distribution function, E_i is the excitation energy, and the indices $i = 1$–4 denote both momentum and spin variables. $W_s(1,2;3,4)$ is the transition probability for excitations 1 and 2 in the superfluid to be scattered to states 3 and 4. At low temperatures the n_i's are exponentially small and the square bracket in (1) becomes n_2. In the superfluid the excitation energies are given in terms of the excitation energy ξ_i in the normal state, as $E_i = (\xi_i^2+\Delta^2)^{1/2}$, and since at low temperature the important values of ξ_i are of order $(\Delta k_BT)^{1/2}$ we may expand E_i in powers of ξ_i:

$$E_i = \Delta + \tfrac{1}{2}\,\xi_i^2/\Delta + \ldots . \quad (2)$$

For the same reason we may evaluate W_s for $\xi_i = 0$. W_s is therefore a function of only two variables, the angles θ and φ used in normal Fermi-liquid calculations.[9] The summations over momenta may be converted into integrals over ξ_i running from minus infinity to plus infinity.[9] We find

$$\tau_1^{-1} = (m^{*3}/8\pi^4\hbar^6)\langle W_s\rangle \int d\xi_2\, n_2 \int d\xi_3 d\xi_4\, \delta((\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2)/2\Delta), \tag{3}$$

where m^* is the effective mass, and

$$\langle W_s\rangle = \int \frac{\sin\theta d\theta d\varphi}{4\pi} \frac{W_s(\theta,\varphi)}{\cos(\theta/2)}, \tag{4}$$

$W_s(\theta,\varphi)$ being the transition probability averaged over initial spin states and summed over final spin states. The integrals over ξ_3 and ξ_4 may be performed directly and give $2\pi\Delta$, and the ξ_2 integral that remains, $\int d\xi_2 n_2$, is just $n_{ex}/\nu(0)$, where

$$n_{ex} = \sum\nolimits_2 n_2 = \nu(0)(2\pi\Delta k_B T)^{1/2} \exp(-\Delta/k_B T)$$

is the number density of excitations, which tends to zero exponentially at low temperatures. [$\nu(0)$ is the density of states of both spins.] Thus one finds

$$\frac{1}{\tau} = \frac{\pi^2}{6}\frac{\Delta}{\hbar}\frac{n_{ex}}{n}[(\hbar/2\pi)\nu(0)^2\langle W_s\rangle]. \tag{5}$$

Here n is the particle number density. The combination in the square brackets is a dimensionless number, which turns out to be ≈ 12 at the melting pressure. Note that the relaxation time is independent of the energy and that we have dropped the subscript on τ_1.

On the basis of simple kinetic arguments one would expect the viscosity η and diffusive thermal conductivity κ_D of the normal component to be given by

$$\eta = \tfrac{1}{5}\rho_n\langle v^2\rangle\tau \tag{6}$$

and

$$\kappa_D = \tfrac{1}{3}c_v\langle v^2\rangle\tau, \tag{7}$$

where

$$\rho_n = (p_F^2/3k_B T)n_{ex} = nm^*(2\pi\Delta/k_B T)^{1/2}\exp(-\Delta/k_B T)$$

is the density of the normal component, $\langle v^2\rangle = (k_B T/\Delta)v_F^2$ is the mean square velocity of an excitation, and $c_v = (\Delta^2/k_B T^2)n_{ex}$ is the heat capacity per unit volume (p_F and v_F are the Fermi momentum and velocity). Inserting Eq. (5) for τ into (6) and (7), one finds

$$\eta = \tfrac{1}{5}nm^* v_F^2(4/\pi^2)(\hbar E_F/\Delta^2)[(\hbar/2\pi)\nu(0)^2\langle W_s\rangle]^{-1}, \tag{8}$$

where $E_F = p_F^2/2m^*$ is the Fermi energy, and

$$\kappa_D = 2\pi^{-2} n v_F^2 \hbar\,[(\hbar/2\pi)\nu(0)^2\langle W_s\rangle]^{-1} T^{-1}. \tag{9}$$

Thus η is a constant of the same order of magnitude as $\eta(T_c)$, the normal-state viscosity at T_c, since $\Delta \sim k_B T_c$. κ_D is independent of Δ and differs from the normal Fermi-liquid result only by a factor of order unity. Qualitatively, this behavior is the same as that found in Refs. 3 and 4.

These results agree *precisely* with those we obtain from exact solutions of the Boltzmann equation. As usual one works with a deviation function ψ_i defined in terms of the deviation from local equilibrium δn_i by $\delta n_i = -(\partial n_i/\partial E_i)\psi_i$. In the superfluid state $\psi \propto \xi$ is an exact solution of the Boltzmann equation, since the excitation velocity v is $(\xi/\Delta)v_F$, and the integral contributions to the collision term are zero for $\psi \propto \xi$.

It is interesting to note that a number of the results above are quite similar to results obtained for rotons in ^{4}He.[10]

To evaluate $\langle W_s\rangle$ we make the usual weak-coupling assumption that the residual interaction in the superfluid state is the same as in the normal state. Thus, the residual interaction between excitations in the superfluid state is found by performing a Bogoliubov transformation on the normal-state interaction. Details of the calculations will be reported elsewhere.[11]

The quasiparticle-scattering amplitude in the normal state may be expressed in terms of its

singlet and triplet components if one neglects the small dipole-dipole interaction. For the simple case when the triplet amplitude vanishes and the singlet amplitude is a constant C_1, we have

$$W_s(\theta,\varphi)=(2\pi/\hbar)\tfrac{1}{32}C_1{}^2\sin^4\tfrac{1}{2}\theta(3+\cos^2\varphi), \quad (10)$$

and $\langle W_s\rangle=(2\pi/\hbar)(7/60)C_1{}^2$. For the same interaction the average of the transition probability in the normal state, $\langle W_N\rangle$, is $\frac{15}{7}\langle W_s\rangle$. The normal-state viscosity may be evaluated exactly,[12] and one finds $\eta(T=0)/\eta(T=T_c)=0.53$ for this interaction if the gap is given by its weak-coupling value. Similarly, one finds $\kappa_D T\simeq 0.87\,\kappa(T_c)T_c$.

To obtain more realistic estimates for liquid ^{3}He we have carried out calculations using for the normal-state scattering amplitude the s- and p-wave approximation which gives a rather good account of the transport coefficients of normal liquid ^{3}He.[13] We find that

$$\langle W_s\rangle=(2\pi/\hbar)\textstyle\sum_{ij}D_{ij}C_iC_j. \quad (11)$$

The C_i are related to the Landau parameters F_l^s and F_l^a by

$$\begin{aligned}&\nu(0)C_1=A_0{}^s-3A_0{}^a, \quad \nu(0)C_2=A_1{}^s-3A_1{}^a,\\ &\nu(0)C_3=A_0{}^s+A_0{}^a, \quad \nu(0)C_4=A_1{}^s+A_1{}^a,\end{aligned} \quad (12)$$

where $A_l{}^i=F_l{}^i[1+F_l{}^i/(2l+1)]^{-1}$. The symmetric matrix D_{ij} has been evaluated numerically with the result

$$D=10^{-3}\times\begin{bmatrix}117 & -8 & -8 & 7\\ -8 & 39 & 21 & -14\\ -8 & 21 & 207 & -45\\ 7 & -14 & -45 & 121\end{bmatrix}. \quad (13)$$

Using the Landau parameters given by Wheatley[14] and $A_1{}^a$ obtained from the forward-scattering sum rule neglecting Landau parameters with $l>1$, one finds $\eta(0)/\eta(T_c)=0.28$ at the melting pressure, 0.27 at 21 bar, 0.25 at 9 bar, and 0.20 at zero pressure if Δ is given by its weak-coupling value. [Strong-coupling effects will increase Δ somewhat and, therefore, reduce $\eta(0)/\eta(T_c)$.] The reason why these ratios are rather smaller than the value 0.53 for the constant s-wave interaction is that the terms involving C_2, C_3, and C_4 increase the normal-state viscosity appreciably compared to its value with the C_1 term alone, but alter the superfluid viscosity only slightly. The theoretical ratio is in rather nice agreement with the data of Alvesalo *et al.*,[1] provided one discards the final upturn around 1.4 mK. When comparing theory and experiment, one should note the following points: First, so far experiments have been made only for $T\gtrsim 0.5T_c$ and finite-temperature contributions to η could possibly be important. Second, near the melting curve the calculated viscosities for both the normal and superfluid phases are about half the measured values, although at zero pressure the calculated and measured normal-state viscosities are in excellent agreement. This may perhaps indicate that the s- and p-wave approximation is inadequate, but we stress the fact that $\eta(0)$, unlike $\eta(T_c)$, is very insensitive to all parameters in the interaction except C_1.

For two-fluid hydrodynamics to be applicable to a vibrating-wire experiment[1] the mean free path l must be less than the viscous penetration depth and the radius of the wire. The mean free path is

$$l=\langle v^2\rangle^{1/2}\,\tau\simeq 0.1\exp(\Delta/k_BT)l_N, \quad (14)$$

where $l_N=v_F\tau_\eta$ is a typical mean free path at T_c (~4 μm at the melting pressure), and τ_η is the viscous relaxation time.

Finally, we remark that the diffusive thermal conductivity will dominate the hydrodynamic flow conductivity[2] κ_H below a certain temperature which depends on the radius of the tube. One finds, for a tube of radius R,

$$\frac{\kappa_D}{\kappa_H}\approx 1\times\frac{10^{-5}}{R^2}\left(\frac{T_c}{T}\right)\exp\left(\frac{2\Delta}{k_BT}\right), \quad (15)$$

if R is measured in centimeters. Thus for $R=0.1$ cm the diffusive heat transport dominates for $T\lesssim 0.5T_c$, and therefore κ_D should be readily measurable with present experimental techniques.

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Nonlinear Wave Number of an Electron Plasma Wave*

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The wave number of an electron plasma wave propagating on a collisionless plasma column is found to differ from that of a small-amplitude wave at the same frequency. The nonlinear wave-number shift oscillates in space. The magnitude of its first maximum, when normalized to the initial (nonlinear) damping coefficient, is found to be proportional to the square root of the initial wave amplitude.

Recent theories[1-3] predict that the wave number, $k_r = 2\pi/\lambda$, of a large-amplitude electron plasma wave is not the same as the wave number of a small-amplitude wave at the same frequency. We have measured[4] the nonlinear wave-number shift using a phase-tracking interferometer. The spatial dependence, magnitude, and scaling with frequency and amplitude are determined.

If the initial wave amplitude, Φ (electric potential), is small enough, the linear theory of Landau[5] applies and is well verified by experiment[6,7]; in a homogeneous plasma a wave of constant frequency has a complex wave number $k = k_r + ik_i$, which is independent of position and Φ. It is also well established[8] in theory and experiment that k_i becomes a function of position and Φ when the wave amplitude is large enough to trap the resonant electrons. Conservation of energy between the wave and the trapped electrons leads to nonlinear amplitude oscillations. The trapped electrons also cause k_r to depend on position and Φ. Conservation of energy in a reference frame moving with the linear-wave phase velocity, v_p, leads to nonlinear oscillations in k_r. In this frame, the amplitude oscillations are related to conservation of momentum.

Morales and O'Neil[1] considered the initial-value problem. A wave of fixed wavelength is turned on at $t = 0$ and propagates in an infinite, homogeneous, one-dimensional, collisionless plasma. The time-dependent shift in the complex frequency caused by the trapped electrons is calculated. They assume that the wave amplitude is constant (negligible linear damping) and small enough for the resonant electrons to be represented by a second-order Taylor expansion of the electron velocity distribution function, $f_0(v)$, about v_p. They also give the transformation to the boundary-value problem studied in our experiment. In this case a wave of fixed frequency ω, determined by a single generator, propagates away from a transmitter probe at $z = 0$. The time-dependent frequency shift becomes a space-dependent shift in the complex wave number. The imaginary part of the shift gives the amplitude oscillations. The real part is

$$\delta k_r(z) = -(\Omega_0/v_g) g(k_B z), \tag{1}$$

where $\Omega_0 = (e\Phi/m)^{1/2}(\omega_p/k_r)^2(\partial^2 f_0/\partial v^2)_{v_p}/(\partial\epsilon/\partial\omega)_{\omega,k_r}$, ϵ is the plasma dielectric function, k_r is the linear wave number, ω is the (constant) oscillation frequency, v_g is the linear group velocity, ω_p is the plasma frequency, and $-e$ and m are the charge and mass of the electron. The function $g(k_B z)$ (Fig. 2 of Ref. 1) gives the space dependence in terms of the scaled position $k_B z$, where $k_B = \omega_B/v_p$ and $\omega_B = (ek_r^2\Phi/m)^{1/2}$ is the bounce frequency of the trapped electrons. If $f_0(v)$ is Maxwellian, $\Omega_0 > 0$. Since $g(k_B z) \leq 0$, Eq. (1) gives $\delta k_r \geq 0$. Examination of $g(k_B z)$ shows that δk_r is initially zero; it increases and executes damped oscillations about an asymptotic value. The os-