Yang's Gravitational Field Equations

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(Received 5 November 1974)

I examine the field equations for the gravitational field *in vacuo* recently proposed by Yang. Similarities with the orthodox theory are pointed out but the equations appear to afford several "unphysical" solutions which will require investigation.

Yang,[1] in his recent discussion of gauge fields, has suggested the identification of the gravitational field equations in relativity theory with the equations for a parallel-displacement gauge field. With this identification a pure gravitational field is to be described by a four-dimensional Lorentz manifold satisfying

$$R_{\alpha\beta;\gamma} - R_{\alpha\gamma;\beta} = 0, \tag{1}$$

for the Ricci tensor $R_{\alpha\beta}$. A similar proposal was also made by Kilmister and Newman.[2] This note will be concerned with those manifolds on which (1) is satisfied, but for which $R_{\alpha\beta} \neq \frac{1}{4} R g_{\alpha\beta}$.

From the Bianchi identities it follows that the scalar curvature $R = R^{\alpha}{}_{\alpha}$ of the manifold is constant, and that the above equations can be written as

$$R^{\alpha}{}_{\beta\gamma\delta;\alpha} = 0, \tag{2}$$

or equivalently

$$C^{\alpha}{}_{\beta\gamma\delta;\alpha} = 0, \quad R = R^{\alpha}{}_{\alpha} = \text{const}, \tag{3}$$

where $R^{\alpha}{}_{\beta\gamma\delta}$ and $C^{\alpha}{}_{\beta\gamma\delta}$ denote the Riemann and Weyl tensors, respectively. Further from (2) with the use of the Ricci identity it follows that the curvature tensor satisfies the quadratic identity[3]

$$R_{\alpha\beta[\gamma\delta} R^{\alpha}{}_{\epsilon]} = 0 \tag{4}$$

(where the square brackets denote antisymmetrization), or equivalently

$$G_{\alpha\beta[\gamma\delta} R^{\alpha}{}_{\epsilon]} = 0.$$

This identity may prove useful in discussing the algebraic properties of the "non-Einsteinian" solutions of (1) (i.e., those for which $R_{\alpha\beta} \neq \frac{1}{4} R g_{\alpha\beta}$). For example, (4) precludes the existence of certain subclasses of Petrov types: types I-2, I-4, Ie-1, Ie-3, as well as those subclasses of types I-1, I-3, and I-3, and I-6, for which none of the eigenvalues of the Weyl tensor are real; cf. the classification of Jordan, Ehlers, and Kundt.[4]

Some specific solutions.—(a) A straightforward computation yields the rather surprising result that the Einstein universe,

$$ds^2 = -(1 - r^2/k^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + dt^2,$$

$k = \text{const}$, is a solution of (1). (b) From (3) we see immediately that all solutions of Littlewood's equations for pure gravitational fields[5] are "pure-space" solutions in Yang's sense. These include the spherically symmetric example

$$ds^2 = -A^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + A^{-2} dt^2,$$

where $A = 1 + k/r$, $k = \text{const}$. The fractional advance of perihelion per revolution for this space has been calculated by Pirani[6]; it is one-sixth of the amount derived from the Schwarzschild solution, and in the opposite direction. (c) A class of axially symmetric static solutions are given by

$$ds^2 = -p^2(dr^2 + r^2 d\theta^2 + dz^2 - dt^2),$$

with $p = p(r, z)$ a solution of Laplace's equation. The choice

$$p = \frac{m}{[r^2 + (z-a)^2]^{1/2}} + \frac{m}{[r^2 + (z+a)^2]^{1/2}}$$

could naively be interpreted as a *static* solution with two particles of equal mass m at points $z < \pm a$ on the axis $r = 0$, without the presence of a "massless rod" holding them apart (cf. Bergmann[7]).

Some similarities with the orthodox Einstein theory.—(a) With use of the metric for the time-dependent spherically symmetric case in the form[8]

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + e^{\nu} dt^2,$$

where $\lambda = \lambda(r, t)$ and $\nu = \nu(r, t)$, the classical *Birkhoff theorem* of general relativity generalizes to the manifolds satisfying (1); viz. "the time dependence of solutions to (1), of the above form, can be removed by a transformation of the time-variable." (b) *The Goldberg-Sachs theorem* relating algebraically special pure gravitational fields and the existence of a congruence of null, shear-free geodesics in vacuum space-times generalizes to all solutions of (1). This is implicit in the conformal generalization of the Goldberg-Sachs theorem given by Kundt and Thompson.[9] (c) It is well known that the necessary and sufficient conditions for a pair of special Einstein spaces (nonflat) to be conformally related is that they be of Petrov type N and that the fourfold principal null vector is hypersurface orthogonal. Modulo the fact that Eqs. (1) admit conformally flat solutions, the above result also holds for the (nonconformally flat) situation discussed here.

These and other results will be derived and discussed in more detail in a forthcoming paper.

[1]C. N. Yang, Phys. Rev. Lett. 33, 445 (1974).

[2]C. W. Kilmister and D. J. Newman, Proc. Cambridge Phil. Soc. 57, 851 (1961).

[3]F. A. E. Pirani, private communication.

[4]P. Jordan, J. Ehlers, and W. Kundt, Acad. Wiss. Mainz 21, No. 2 (1960).

[5]D. E. Littlewood, Proc. Cambridge Phil. Soc. 49, 90 (1953).

[6]F. A. E. Pirani, Proc. Cambridge Phil. Soc. 51, 535 (1955).

[7]P. Bergman, *Introduction to the Theory of Relativity* (Prentice-Hall, Englewood Cliffs, N. J., 1946).

[8]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, Oxford, England, 1950).

[9]W. Kundt and A. H. Thompson, C. R. Acad. Sci. 254, 4251 (1962).

ERRATA

SPECULATIONS ON DETECTION OF THE "NEUTRINO SEA." L. Stodolsky [Phys. Rev. Lett. 34, 110 (1975)].

P. R. Phillips [Phys. Rev. 180, 1331 (1969)] and P. R. Phillips and D. Woolum [Nuovo Cimento 64B, 28 (1969)] have discussed and investigated the effects of a vector field in space, of which my "neutrino sea" would be one example. Some of the effects are quite similar to those I discussed. The experimental accuracy reached, of course, is far from that needed for the neutrino sea according to my estimates.

On page 110, first column, fifth line from the bottom, the word "neutral" was omitted. The text should read "(If the neutral currents exist . . .)."

REMARKS ON THE 3105-MeV RESONANCE. T. N. Pham, B. Pire, and Tran N. Truong [Phys. Rev. Lett. 34, 347 (1975)].

Equation (3) should read

$$\frac{1}{2M\Delta'}\int_{(M-\Delta'/2)^2}^{(M+\Delta'/2)^2}\left(\frac{f_{\gamma Y}}{M^2}\right)^4 M^4 \times\frac{ds}{(s-M^2)^2+M^2\Gamma^2}\simeq 0.7.$$