and $\mathcal{E}$ is the optical pulse energy of strong beam. The quantity $G(\tau)$ is the electric field autocorrelation function. Substituting the experimental parameters $m^* = 0.088m_0$, $\mathcal{E} = 113$ J/m^2, $l = 5\times10^{-6}$ m^{-1}, we obtain $a = 2.5$. The peak-to-background ratio is then

$$T(\tau=0)/T(\tau=\infty) = 3.5.$$

This is somewhat larger than the experimental value of approximately 2.

We would expect the measured spike to be somewhat broader and reduced in amplitude from that predicted by the electric field autocorrelation function because of imperfect overlap of the two beams. A bandwidth of 10 cm^{-1} would produce a spike of width of approximately 1 psec.

We believe the slower feature in the experiment (Fig. 1) to be due to a saturation of the absorption by band filling, i.e., a filling of conduction-band states and depletion of valence-band states to the point where the separation between electron and hole quasi-Fermi levels approaches the photon energy. The buildup of this effect follows the integrated optical pulse energy since recombination is expected to be slow. The decay, however, is most likely due to a reduction in density by the diffusion of the electron and holes from a region approximately α^{-1} (1 μm) near the surface into the crystal (approximately 5 μm thick). While the dynamics of this decay are quite complex, we can see that the overall decay rate is comparable to previous measurements[2] of diffusion rates. We can make a rough estimate of the plasma density required to produce band filling by calculating the positions of the electron and hole quasi-Fermi levels assuming parabolic bands. Using the density-of-states effective masses, $m_e = 0.55m_0$ and $m_h = 0.35m_0$, we find that the quasi-Fermi levels are separated by the photon energy of 1.17 eV (compared to $E_g = 0.66$ eV) at a density of approximately 2×10^{20} cm^{-3}, slightly less than the estimated experimental value. Although band filling could also contribute to the parametric coupling responsible for the spike by an amplitude grating, this process is at least an order of magnitude smaller than the plasma index mechanism.

In conclusion we have demonstrated that parametric coupling in an electron-hole plasma can explain the results of Kennedy *et al.* without requiring an ultrafast relaxation process. Furthermore, we have observed a true saturation of the absorption which can be accounted for by band filling.

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Physical Realizations of $n \geqslant 4$ Vector Models*

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It is pointed out that certain phase transitions which involve a doubling of the unit cell are described by n-component vector models with $n \geqslant 4$. In particular, it is noted that the structural transition in NbO_2 is described by an $n = 4$ component model with some tetragonal anisotropy. The critical behavior of this model is studied to order ϵ^2, by the exact renormalization group in $d = 4-\epsilon$ dimensions. It is found that the critical behavior is determined by a new, tetragonal, fixed point.

The critical behavior of the n-component vector model has been of considerable interest in recent years.[1-7] For $n = 1, 2, 3$ the model corresponds to physical systems which are Ising, X-Y, and Heisenberg-like, respectively. It has also been argued[8,9] that the limit $n \to 0$ corresponds to amorphous Ising systems. In this work, it is pointed out that certain phase transitions, which involve a doubling of the unit cell in one or more directions, are described by n-component models with $n \geqslant 4$. As discussed by Landau,[10] the symmetry-breaking order parameter associated with a second-order phase transition transforms as an irreducible representation of the symmetry group of the high-symmetry phase. The number of independent components of the order parameter is, therefore, equal to the dimensionality n of the representation according to which it transforms, and the transition is described by an n-component model. For transitions which do not involve a change of the unit cell, the dimensionality n satisfies $n \leq 3$. The reason is that in these cases the order parameter transforms as an irreducible representation of the *point group* of the high-symmetry phase, and the dimensionality of these representations[11] is smaller than 4. In case that the unit cell is doubled in one or more directions, the relevant group is the *space group*, and, in principle, it may have irreducible representations with dimensionality $n \geq 4$. There exist many systems which exhibit transitions associated with highly degenerate representations. The structural phase transition in NbO_2[12] corresponds to $n=4$, the antiferromagnetic transitions in ErSb,[13] MnSe,[14] and α-MnS[15] correspond to $n=8$, and there exist other examples. It would be interesting to compare the critical behavior of these systems with calculations done on the appropriate n-vector models. The study of the critical behavior of these systems may also provide an experimental test for the regions of validity of the $1/n$ and ϵ expansions.[3,5]

In this work I discuss in detail the structural phase transition in NbO_2. It is shown that this transition is described by an $n=4$ vector model with some tetragonal anisotropy. The critical behavior of this model is studied in the exact renormalization group to second order in ϵ. To do that, one should write a Landau-Wilson-type Hamiltonian to fourth order in the order parameter, which is invariant under the group of the high-symmetry phase. In general, one obtains an anisotropic n vector model whose anisotropy is determined by the symmetry of the problem and the representation involved in the transition. I will show that the appropriate Hamiltonian for describing the phase transition in NbO_2 is given by

$$\mathcal{H}=\int dV\{-\tfrac{1}{2}r\sum_{i=1}^{4}\varphi_i^2-\tfrac{1}{2}\sum_{i=1}^{4}(\nabla\varphi_i)^2-u(\sum_{i=1}^{4}\varphi_i^2)^2-v\sum_{i=1}^{4}\varphi_i^4-w(\varphi_1^2\varphi_2^2+\varphi_3^2\varphi_4^2)\}. \quad (1)$$

This Hamiltonian has two nonisotropic terms: v, which has cubic symmetry, and w, which has tetragonal symmetry (see the following section). The critical behavior of this model with $w=0$ has been studied for arbitrary n to second order in ϵ by Aharony.[7] For $n=4$ he found that the stable fixed point to lowest nontrivial order in ϵ is the cubic point. In studying the Hamiltonian (1), I find that the cubic fixed point is unstable to w perturbations. The model is found to have two tetragonal fixed points, one of them is stable to lowest nontrivial order in ϵ. Taking $\epsilon=1$ one finds, for example, that the critical exponent β which corresponds to the stable fixed point is given by $\beta=0.39$. This exponent is now being measured at Brookhaven National Laboratory.

NbO_2 undergoes a second-order structural phase transition at $T_c\sim 800°C$, in which the symmetry is reduced from $P4_2/mnm$, in the high-temperature phase, to $I4_1/a$ in the low-temperature phase. Neutron-diffraction[12] as well as x-ray measurements[16] show that the unit cell below the transition is 16 times larger than that above the transition, and that the order parameter which becomes critical at T_c belongs to a reciprocal-lattice vector $k=(\frac{1}{4},\frac{1}{4},\frac{1}{2})$. The little group of this vector is $2mm$. The order parameter belongs to one irreducible representation of the little group, and preliminary studies[17] show that this representation is either A_1 or A_2 (both are one-dimensional).[11] The star of the vector k consists of four vectors, $\pm k$ and $\pm\bar{k}$, where $\bar{k}=(\frac{1}{4},-\frac{1}{4},\frac{1}{2})$. The order parameter which is associated with the transition has, therefore, four independent components, which can be written as

$$\Psi_k\equiv\varphi_1+i\varphi_2,\quad \Psi_{-k}\equiv\varphi_1-i\varphi_2,$$
$$\Psi_{\bar{k}}\equiv\varphi_3+i\varphi_4,\quad \Psi_{-\bar{k}}\equiv\varphi_3-i\varphi_4, \quad (2)$$

where φ_l, $l=1,\ldots,4$, are real parameters. The order parameter (2) has one second-order invariant

$$\Psi_k\Psi_{-k}+\Psi_{\bar{k}}\Psi_{-\bar{k}}\equiv\sum_{i=1}^{4}\varphi_i^2, \quad (3)$$

and three fourth-order invariants

$$(\Psi_k\Psi_{-k}+\Psi_{\bar{k}}\Psi_{-\bar{k}})^2\equiv(\sum_{i=1}^{4}\varphi_i^2)^2, \quad (4a)$$

$$\Psi_k\Psi_{-k}\Psi_{\bar{k}}\Psi_{-\bar{k}}\equiv\varphi_1^2\varphi_3^2+\varphi_1^2\varphi_4^2+\varphi_2^2\varphi_3^2+\varphi_2^2\varphi_4^2, \quad (4b)$$

$$\Psi_k^4+\Psi_{-k}^4+\Psi_{\bar{k}}^4+\Psi_{-\bar{k}}^4\equiv 2\sum_{i=1}^{4}\varphi_i^4-12(\varphi_1^2\varphi_2^2+\varphi_3^2\varphi_4^2). \quad (4c)$$

The Hamiltonian of this system, to fourth order in φ_i, is therefore given by[18] (1). We assume in this work that the exchange is isotropic and is given by

$$\mathcal{H}_{ex} = \sum_{i=1}^{4} (\Delta\varphi_i)^2. \tag{5a}$$

In general, the exchange which is compatible with the tetragonal symmetry is given by[19]

$$\mathcal{H}_{ex} = \sum_{i=1}^{2}\left[\left(\frac{\partial\varphi_i}{\partial z}\right)^2 + a\left(\frac{\partial\varphi_i}{\partial x} + \frac{\partial\varphi_i}{\partial y}\right)^2 + b\left(\frac{\partial\varphi_i}{\partial x} - \frac{\partial\varphi_i}{\partial y}\right)^2\right] + \sum_{j=3}^{4}\left[\left(\frac{\partial\varphi_j}{\partial z}\right)^2 + b\left(\frac{\partial\varphi_j}{\partial x} - \frac{\partial\varphi_j}{\partial y}\right)^2 + a\left(\frac{\partial\varphi_j}{\partial x} - \frac{\partial\varphi_j}{\partial y}\right)^2\right]. \tag{5b}$$

Equation (5b) is reduced to (5a) when $a = b = 1$. It has been shown by Bruce[20] that in case of cubic symmetry, the anisotropic exchange is an irrelevant variable in the ϵ expansion. We assume that the anisotropic exchange is an irrelevant variable in our case too, and discuss the critical behavior of the Hamiltonian (1) with the isotropic exchange.

The Hamiltonian (1) describes transitions which correspond not only to the reciprocal-lattice vector $k = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, but to a more general class of reciprocal-lattice vectors, $k = (\zeta, \zeta, 0)$ and $k = (\zeta, \zeta, \frac{1}{2})$, where $0 < \zeta < 1$. For $\zeta \neq \frac{1}{4}$, the term (4c) is no longer an invariant under the space group $P4_2/mnm$, and there exist only two independent fourth-order invariants (4a) and (4b). For such a transition, the system will be described by the Hamiltonian (1) with $w = 2v$. It is found that the fixed point of the Hamiltonian (1) which is stable to lowest nontrivial order in ϵ satisfies $w^* = 2v^*$. This point may, therefore, describe the critical behavior of transitions associated with the more general class of reciprocal-lattice vectors discussed above.

Using the ϵ-expansion technique introduced by Wilson and Fisher,[3,21] and assuming a small momentum cutoff b^{-1}, one may keep only terms of order $\ln b$ in the recursion relations for u', v', w', and terms of order $b^2 \ln b$ in the recursion relations for r'. The following recursion relations to order ϵ^2 are obtained:

$$r' = b^{2-\eta}[r + (24u + 12v + 2w)A(r) - (192u^2 + 96v^2 + 192uv + 8w^2 + 32uw)B(r)] + O(\epsilon^3),$$

$$u' = b^{\epsilon-2\eta}[u - 48u^2 + 24uv + 4uw)K_d \ln b + (21\times 2^6u^3 + 9\times 2^7u^2v + 9\times 2^5uv^2 + 3\times 2^6u^2w + 24w^2u)K_4{}^2 \ln b] + O(\epsilon^4),$$

$$v' = b^{\epsilon-2\eta}[v - (36v^2 + 48uv + w^2)K_d \ln b + (27\times 2^6u^2v + 9\times 2^8uv^2 + 27\times 2^5v^3 + 48uw^2 + 96uvw + 24w^2v + 8w^3)K_4{}^2 \ln b] + O(\epsilon^4),$$

$$w' = b^{\epsilon-2\eta}[w - (48uw + 24vw + 8w^2)K_d \ln b + (27\times 2^6u^2w + 15\times 2^5w^2u + 27\times 2^6uvw + 9\times 2^5w^2v + 9\times 2^5wv^2 + 40w^3)K_4{}^2 \ln b] + O(\epsilon^4), \tag{6}$$

where

$$\eta = (48u^2 + 48uv + 24v^2 + 2w^2 + 8uw)K_4{}^2 + O(\epsilon^3), \tag{7}$$

$A(r)$ and $B(r)$ are the integrals over the propagator, and K_d is the angular integral in d dimensions.[3,21] The recursion relations (6) have eight fixed points. The u^*, v^*, w^*, and the eigenvalue exponents λ_i, $i = 1, 2, 3$, for each fixed point are given in Table I. (The exponents are denoted by λ_u, λ_v, λ_w when the eigenvectors are u, v, and w, respectively.) The λ_i's are obtained by linearizing the recursion relations (6) and diagonalizing the linearized relations. It is clear from Table I that the only fixed point which is stable (corresponds to negative λ_i's) to lowest nontrivial order in ϵ is the tetragonal point II (point No. 8). Also, this is the only point which is stable at $\epsilon = 1$. All the other points, except point No. 4, are unstable for $\epsilon \leq 1$. Point No. 4 has a marginal operator at $\epsilon = 1$ (corresponds to the eigenvalue $\lambda_1 = 0$), and is unstable for $\epsilon < 1$. The stability of this point at $\epsilon = 1$ should be determined by the next order term in ϵ.

The critical exponent ν is given by

$$1/\nu = 2 + 5\eta - (24u^* + 12v^* + 2w^*)K_d + O(\epsilon^3). \tag{8}$$

Equations (7) and (8) give, for the tetragonal point,

$$\nu = \tfrac{1}{2}(1 + \tfrac{1}{4}\epsilon + \tfrac{7}{48}\epsilon^2) + O(\epsilon^3), \tag{9a}$$

$$\eta = \tfrac{1}{48}\epsilon^2 + O(\epsilon^3). \tag{9b}$$

These exponents are equal to those of the $n = 4$ isotropic fixed point (but they may be different in higher order in ϵ). For $\epsilon = 1$ one obtains

$$\nu = 0.70, \quad \eta = 0.02 \tag{10a}$$

TABLE I. The fixed points and the eigenvalue exponents of the Hamiltonian (1) to second order in ϵ.

No	Fixed Point	Type	Eigenvalues
1.	$u^* = v^* = w^* = 0$	Gaussian	$\lambda_u = \lambda_v = \lambda_w = \varepsilon$
2.	$u^* = w^* = 0;\ v^* = \frac{\varepsilon}{36K_d} + \frac{17}{36\cdot 27K_4}\varepsilon^2$	Ising	$\lambda_u = \lambda_w = \frac{1}{3}\varepsilon - \frac{19}{81}\varepsilon^2 \quad \lambda_v = -\varepsilon + \frac{17}{27}\varepsilon^2$
3.	$u^* = 0 \quad v^* = \frac{\varepsilon}{72K_d} + \frac{17}{72\cdot 27K_4}\varepsilon^2$ $w^* = \frac{\varepsilon}{12K_d} + \frac{17}{9\cdot 36K_4}\varepsilon^2$	n=2 Cubic	$\lambda_u = \lambda_1 = \frac{1}{3}\varepsilon - \frac{19}{81}\varepsilon^2 \quad \lambda_2 = -\varepsilon + \frac{17}{27}\varepsilon^2$
4.	$u^* = 0 \quad v^* = \frac{\varepsilon}{40K_d} + \frac{3}{200K_4}\varepsilon^2$ $w^* = \frac{\varepsilon}{20K_d} + \frac{3}{100K_4}\varepsilon^2$	n=2 Isotropic	$\lambda_u = \frac{1}{5}\varepsilon - \frac{28}{100}\varepsilon^2 \quad \lambda_1 = -\frac{1}{5}\varepsilon + \frac{1}{5}\varepsilon^2$ $\lambda_2 = -\varepsilon + \frac{3}{5}\varepsilon^2$
5.	$w^* = v^* = 0 \quad u^* = \frac{\varepsilon}{48K_d} + \frac{26}{48^2K_4}\varepsilon^2$	n=4 Isotropic	$\lambda_u = -\varepsilon + \frac{13}{24}\varepsilon^2 \quad \lambda_v = \lambda_w = \frac{1}{6}\varepsilon^2$
6.	$w^* = 0 \quad u^* = \frac{\varepsilon}{48K_d} + \frac{10}{48^2K_4}\varepsilon^2$ $v^* = \frac{\varepsilon^2}{72K_4}$	n=4 Cubic	$\lambda_w = \frac{1}{6}\varepsilon^2 \quad \lambda_1 = -\varepsilon + \frac{13}{24}\varepsilon^2$ $\lambda_2 = -\frac{1}{6}\varepsilon^2$
7.	$w^* = \frac{1}{24K_4}\varepsilon^2\ ;\ v^* = \frac{1}{48\cdot 3K_4}\varepsilon^2$ $u^* = \frac{\varepsilon}{48K_d} + \frac{10}{48^2K_4}\varepsilon^2$	n=4 Tetragonal I	$\lambda_1 = \frac{1}{6}\varepsilon^2 \quad \lambda_2 = -\frac{1}{6}\varepsilon^2 \quad \lambda_3 = -\varepsilon + \frac{13}{24}\varepsilon^2$
8.	$w^* = \frac{1}{24K_4}\varepsilon^2;\ v^* = \frac{1}{48K_4}\varepsilon^2$ $u^* = \frac{1}{48K_d}\varepsilon - \frac{1}{48\cdot 8K_4}\varepsilon^2$	n=4 Tetragonal II	$\lambda_1 = \lambda_2 = -\frac{1}{6}\varepsilon^2 \quad \lambda_3 = -\varepsilon + \frac{13}{24}\varepsilon^2$

and using scaling relations

$$\beta = 0.39, \quad \gamma = 1.39, \quad \alpha = -0.17, \quad \delta = 4.46. \quad (10b)$$

If one calculates the λ_i's to higher order in ϵ, one may find that the tetragonal point II is unstable at $\epsilon = 1$, and that the critical behavior is determined by some other fixed point. It would thus be interesting to determine the critical exponents experimentally, and compare them with the calculated results.

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Note added.—After the completion of this work, I have learned of a work by Alben,[22] where he discussed, within the Landau theory, the existence of $n = 4$- or 8-dimensional order parameters in type-II antiferromagnets. I would like to thank Professor Alben for sending me his reprint.

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$$-\tfrac{1}{2}C\left(\varphi_1\frac{\partial\varphi_2}{\partial\alpha}-\varphi_2\frac{\partial\varphi_1}{\partial\alpha}+\varphi_3\frac{\partial\varphi_4}{\partial\beta}-\varphi_4\frac{\partial\varphi_3}{\partial\beta}\right),$$

where $\alpha=x+y$ and $\beta=x-y$. This term will not affect the critical behavior of the system. We shall discuss that in a later publication.
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Detection of He Ions in the Interaction of 70-MeV π^- with Aluminum*

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We detected and identified He ions, predominantly α particles, in the reaction $\pi^- + {}^{27}\mathrm{Al} \rightarrow \mathrm{He}+X$ at $E_\pi=70$ MeV. The differential cross section at 90° is 6±2 mb/sr in the energy range $E_{\mathrm{He}}=5.5$–30 MeV. The He yield decreases from 7 to 25 MeV, where it approaches zero.

In several recent nuclear reactions with pions,[1-4] kaons,[5] and protons,[6,7] the removal of one or more "α" clusters is strongly observed. Several theoretical suggestions[8,9] have been made to describe the mechanisms for the strong emission of α particles in pion-nucleus reactions. In most of the experiments with pions and kaons, prompt γ rays from residual nuclei formed in the reactions were detected. The "removed equivalent cluster" was not directly identified, and was either single nucleons or heavier particles. Moreover, by this γ-ray method, reactions leading directly to the ground state are not observed. Some of these cases have been observed[10] via activation methods. However, α emission often leads to stable residual nuclei and in these cases this channel cannot be observed. Castleberry *et al.*,[11] working with stopped negative pions, observed charged-particle emission but did not see He ions. Gismatulin, Ostrumov, and Plyushchev[12] observed α tracks from light emulsion nuclei at $E_{\pi^+}=112$ MeV, with poor statistics, and found a relatively low yield for α emission. On the other hand, Ashery *et al.*[1] found strong lines corresponding to "α"-cluster removal at $E_\pi=25$ MeV, which is below the threshold for the knockout of four individual nucleons. However, this can be taken as a proof of α emission only on the assumption that the pions were not absorbed. Thus, to date, recent reports on the strong emission of "α" clusters in the interaction of pions with nuclei were based on assumptions. Our primary purpose here was to provide a reliable direct proof for α-particle emission in pion-nucleus interaction, and to measure their energy spectrum and cross section.

Our report here describes the detection and identification of He ions emitted in the reaction $\pi^- + {}^{27}\mathrm{Al} \rightarrow \mathrm{He}+X$, thus providing a direct observation of emitted He ions and of their energy spectrum. We used the secondary π^- beam from the electron linear accelerator at Centre d'Etudes Nucléaires de Saclay. The energy of the pions was 70 MeV, with $\Delta p/p=5\%$. The beam intensity