# Critical Exponents for the Shear Viscosity of $^4$He at $T_\lambda$*

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Precision measurements of the viscosity ($\eta$) of liquid $^4$He at the saturated vapor pressure over the range $2.0\times10^{-6}\lesssim|\epsilon|\lesssim5.5\times10^{-3}$ ($\epsilon=T/T_\lambda-1$) have revealed a singularity of the form $\eta/\eta_\lambda-1=-A'(-\epsilon)^{0.65\pm0.03}$ below $T_\lambda$ and $\eta/\eta_\lambda-1=A\epsilon^{0.80\pm0.05}$ above $T_\lambda$.

For many years there has been a question as to what singularity, if any, occurs in the viscosity of liquid $^4$He at its superfluid transition ($T_\lambda$). A definitive theoretical prediction has not yet been made and definitive measurements have proved elusive.[1] We report here a detailed measurement of the singularity, and we determine the critical exponents above and below $T_\lambda$. Earlier experiments give results which are consistent with a weak singularity,[2] but they are unable to determine what happens when $|T-T_\lambda|\lesssim10^{-4}$ K, what functional form gives an adequate description of $\eta(T)$, and how the behavior of $\eta$ above $T_\lambda$ compares with that below. We find that for $T$ sufficiently near $T_\lambda$, $\eta$ is fitted by the forms

$$\eta/\eta_\lambda-1=-(1.3\pm0.04)(-\epsilon)^{0.65\pm0.03}$$

below $T_\lambda$, and

$$\eta/\eta_\lambda-1=(2.1\pm0.9)\epsilon^{0.80\pm0.05}$$

above $T_\lambda$. These are the viscosity critical exponents.[3]

Recently, Goodstein, Savoia, and Scaramuzzi[4] have studied the mobility, $\mu$, of ions for $10^{-5}\leq-\epsilon\leq10^{-2}$ and they find that $\mu/\mu_\lambda-1\sim(-\epsilon)^{0.96\pm0.02}$ along the saturated-vapor-pressure curve. They point out that if one treats the ion as a sphere of radius $R\approx10$ Å in a viscous fluid and applies Stokes law, then $\mu=e(6\pi R\eta)^{-1}$, and so $\eta$ would also have a critical exponent of 0.96. One can resolve the discrepancy between this result and our measurements by discarding the simple hydrodynamic relation between $\mu$ and $\eta$ on the grounds that it does not apply because the correlation length (estimated as $\sim\epsilon^{-2/3}$ Å) for fluctuations in the superfluid order parameter was larger than the 10-Å ion diameter.

Our technique is to measure the damping on a torsion pendulum, whose hollow, cylindrical bob contains the helium. The cylinder is made of copper and has an inside height of 0.7 cm and an inside radius of 2.34 cm. The normal fluid behaves as a classical, viscous fluid, which we assume has zero slip at the walls. Thus, the oscillating walls produce a shear velocity field which falls off as $\exp(-r'/\delta)$, where $r'$ is the distance from the wall and $\delta=(2\eta/\rho_n\omega)^{1/2}$ ($\sim2.8$ μm in our case).[5] We require thermal equilibrium only over a layer of thickness $\sim3\delta$ and not over the whole sample. Since at our smallest $|\epsilon|$, $\delta$ is still 5 times the correlation length estimated above, we are justified in using classical hydrodynamics to analyze this system.

The torque on the pendulum due to the fluid has real and imaginary parts, each proportional to $(\eta\rho_n)^{1/2}$. The former gives rise to viscous damping, while the latter makes a negligible contribution to the moment of inertia. At a boundary, the normal-fluid fraction is expected to go to 1 over a characteristic "healing length."[6] When the viscous penetration depth ($\delta$) is less than or equal to this length, a significant correction must be made to the relation between torque and viscosity given above. Our data just avoid this regime. An important feature of our apparatus is that the peak angular velocity of the oscillation can be as small as 252 μrad/sec, which at the 732-Hz resonant frequency corresponds to a wall velocity of $5.9\times10^{-4}$ cm/sec and a wall azimuthal displacement of 12.8 Å. These velocities are below any known critical velocity for vortex creation or the appearance of nonlinear effects.

A coil, attached to the bottom of the copper sample chamber and positioned in the gap of a permanent magnet, drives the pendulum at resonance. Its effective impedance is resistive and inversely proportional to the damping coefficient. The pendulum-magnet assembly hangs on a hollow vibration-isolation spring in a vacuum can, which is surrounded by a temperature-regulated helium bath. An Allen-Bradley carbon resistor is mounted on the sample chamber to serve as a thermometer. Its resistance is determined from the off-balance voltage of an ac Wheatstone bridge, which is balanced at $T_\lambda$. We can resolve $\sim1.5$-μK changes in temperature with a 3-sec time con-

stant. The total average error in $|T - T_\lambda|$ is about 5%.

A feedback circuit keeps the pendulum running at a given amplitude on resonance while two lock-in amplifiers monitor the coil voltage, $V_B$ (which is proportional to the angular velocity), and a drive voltage, $V_A$. The driving torque is proportional to $V_A - V_B$. These voltages determine the total damping coefficient, $\alpha$, which is one-half the ratio of the damping torque to the angular momentum. The precision in $\eta/\eta_\lambda$ is determined largely by amplifier drifts and falls between 0.03% (at $|T - T_\lambda| \sim 10$ $\mu$K) and ~0.50% (at $|T - T_\lambda| \sim 2$ mK).

Our procedure is to record the data continuously with the sample warming via exchange-gas contact to the main bath, and with the pendulum running at constant amplitude. Warming of the main bath is controlled manually via a metering valve between it and the vacuum pump. Two $x$-$y$ recorders produce plots of $V_A$ and $V_B$ versus the thermometer voltage ($V_T$). Since it is difficult to achieve smooth temperature control while cooling, data are taken this way only as a check.

Viscous heating in the cell is $\sim 10^{-14}$ W. The damping coefficient for no sample ($\alpha_0$), which is ~40% of the total damping coefficient ($\alpha$), was measured as a function of temperature and subtracted from $\alpha$ to give a value ($\alpha_{He}$) for the helium alone. In each warming session, the $\lambda$ point of the sample was identified to within 3 to 7 $\mu$K by a sudden change in slope of the $V_A$-versus-$V_T$ recording. It was found that $V_T(T_\lambda)$ drifted, implying an increase in $R_\lambda$ with time. The apparent shift in $T_\lambda$ was about 45 to 50 $\mu$K/h. Correction for this effect does not produce any significant change in the results. The data were independent of cell warming rates $<100$ $\mu$K/min far from $T_\lambda$ and $\lesssim 50$ $\mu$K/min within ~500 $\mu$K of $T_\lambda$.

Values of $\eta\rho_n/\eta_\lambda\rho_\lambda$ were computed directly from the damping data at selected temperatures by using the formula

$$\frac{\eta\rho_n}{(\eta\rho)_\lambda} = \left(\frac{\mu(\Delta T) - \mu_0(\Delta T)}{\mu(0) - \mu_0(0)}\right)^2,$$

where $\Delta T \equiv T - T_\lambda$, $\mu = V_A/V_B$ with sample, and $\mu_0 = V_A/V_B$ without sample. The values of $\eta/\eta_\lambda$ then were computed by using the form for $\rho_s/\rho$ given by Greywall and Ahlers[7] and that for $\rho/\rho_\lambda$ by Kerr and Taylor.[8] The results of six runs at a peak angular velocity of 1.26 mrad/sec are shown in Fig. 1. Our current best value for $\eta_\lambda$ is $27.4 \pm 1.4$ $\mu$P.

Figure 2 shows the temperature coefficient of

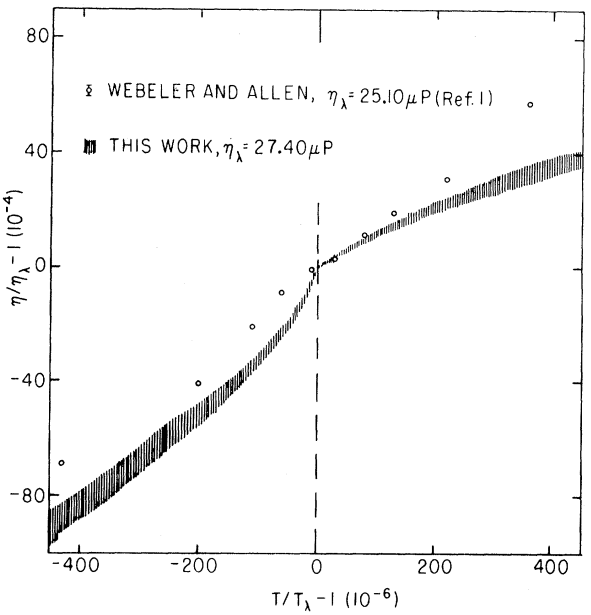


FIG. 1. Reduced viscosity $\eta/\eta_\lambda - 1$ versus reduced temperature $T/T_\lambda - 1$; the results of six runs at 1.26 mrad/sec lie within the hatched region. The critical exponents quoted in the text were determined from data within this temperature range. The Webeler and Allen points are for an 0.5% solution of $^3$He in $^4$He. The origin of the discrepancy between their data and ours is not known at this time.

viscosity, $\eta^{-1}d\eta/dT$, as derived from the data in Fig. 1 and additional data at 0.25, 2.52, 6.30, and 18.9 mrad/sec. This was computed at each $\epsilon$ by fitting a parabola to five neighboring points of the function $\zeta \equiv \eta/\eta_\lambda^{-1}$ and then using the relation

$$\frac{1}{\eta}\frac{d\eta}{dT} = \frac{1}{(1+\zeta)T_\lambda}\frac{d\zeta}{d\epsilon},$$

where $d\zeta/d\epsilon$ is the slope of the parabola at the midpoint. Note that as $T \to T_\lambda$ from below, $\eta^{-1} \times d\eta/dT$ rises sharply, especially in the last 50 $\mu$K. No such divergence is apparent above $T_\lambda$, but we cannot rule it out in the last 15 $\mu$K. The solid lines in Fig. 2 are derived from the logarithmic fits described below. These imply divergence of $\eta^{-1}d\eta/dT$ both above and below $T_\lambda$.

We used the following procedure to determine the best form for $\eta(T)$. The basic $\eta\rho$ data for each run were plotted as $\log|\eta\rho_n/\eta_\lambda\rho_\lambda - 1|$ versus $\log|\epsilon|$ both above and below $T_\lambda$. These plots were approximately linear for $|\epsilon| \leq 4\times10^{-4}$, and so for each run straight lines were fitted separately above and below $T_\lambda$ and their standard errors computed. Then $T_\lambda$ and $\eta_\lambda\rho_\lambda$ were varied within the limits of experimental error to find the val-

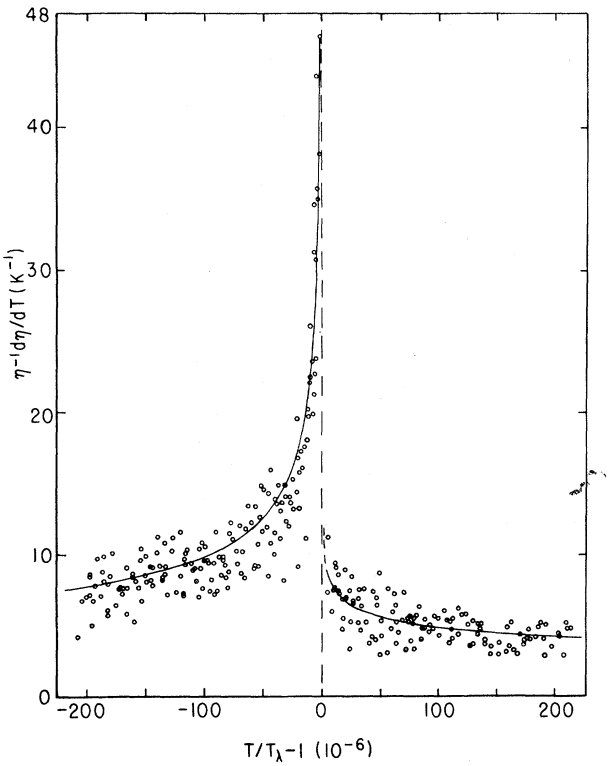


FIG. 2. Temperature coefficient of viscosity, $\eta^{-1}d\eta/dT$, versus reduced temperature. The results of ten runs with peak angular velocities from 0.25 to 18.9 mrad/sec are shown. The solid lines are derived from the power-law dependence given in the text. Note that the temperature scale here is expanded over that of Fig. 1.

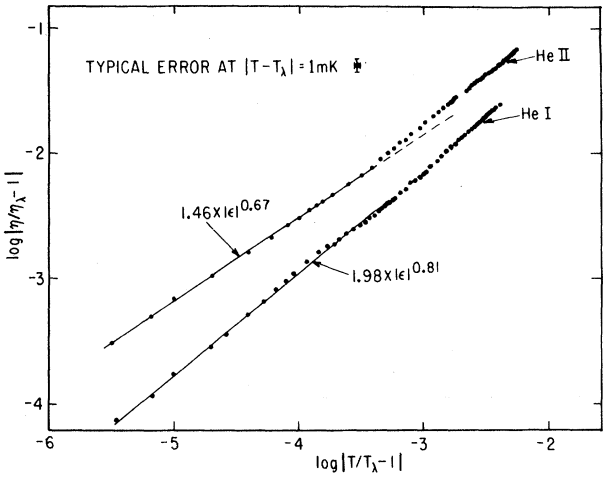


FIG. 3. Reduced viscosity versus reduced temperature on log-log scales for a typical run at 1.26 mrad/sec. The error given is the estimated systematic error. The solid lines have been fitted as described in the text. Note the deviation at large $\epsilon$.

ues which minimized the errors of the slopes.

Averaging the results of ten runs having peak angular velocities from 0.25 to 18.9 mrad/sec, we find that

$$\frac{\eta\rho_n}{\eta_\lambda\rho_\lambda} - 1 = \begin{cases} -(3.7\pm0.4)(-\epsilon)^{0.66\pm0.01}, & T<T_\lambda, \\ (2.1\pm0.9)\epsilon^{0.80\pm0.05}, & T>T_\lambda \end{cases}$$

and

$$\frac{\eta}{\eta_\lambda} - 1 = \begin{cases} -(1.3\pm0.4)(-\epsilon)^{0.65\pm0.03}, & T<T_\lambda, \\ (2.1\pm0.9)\epsilon^{0.80\pm0.05}, & T>T_\lambda. \end{cases}$$

In Fig. 3 the viscosity data of a typical run at 1.26 mrad/sec are shown on log-log scales. The solid line is the best fit for this run. For $|\epsilon| \geq 4\times10^{-4}$ and below $T_\lambda$, the data deviate from the above form and approach the power-law fit by Ahlers[2] to the data of earlier experiments.[1]

We know of no theoretical prediction for the form of $\eta(T_\lambda)$ near $T_\lambda$; however, $\eta(T_\lambda)$ is expected to be finite.[9] Any divergence must occur in the first- or higher-order derivatives. Our results imply a vertical inflection in $\eta(T)$ at $T_\lambda$. We do not know whether there is special significance to the fact that $\Delta\eta/\eta_\lambda$ for $T<T_\lambda$ has the same critical exponent as the superfluid density.

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# Measurement of the Thermal Boundary Resistance between Solid $^3$He and Cerium Magnesium Nitrate

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The thermal boundary resistance between powdered cerium magnesium nitrate and solid $^3$He was measured in the temperature range between 45 and 250 mK. The phonon bottleneck as well as the phonon-dependent boundary resistance was observed. At low temperatures we observed a magnetic Kapitza resistance in parallel with that due to the phonons. The magnetic Kapitza resistance is proportional to $T^2$ in agreement with theoretical predictions.

We have measured the thermal boundary resistance between powdered cerium magnesium nitrate (CMN) (average particle diameter ~50 $\mu$m) and solid $^3$He (23.9 cm$^3$/mole, ~15 ppm $^4$He) at several different applied magnetic field strengths in the temperature range between 45 and 250 mK. At temperatures below 70 mK and fields greater than ~55 G, the observed measurements of the magnetic Kapitza resistance are consistent with the $T^2$ dependence of that resistance predicted by Guyer.[1]

The observation of anomalously short time constants for thermal equilibrium between liquid $^3$He and powdered CMN was first reported in 1966 by Able *et al.*[2] At a CMN magnetic temperature of 2 mK, Black *et al.*[3] observed that this time constant was about 100 times smaller than it would have been if the rate of energy transfer was limited by the usual phonon boundary resistance.

Leggett and Vuorio[4] proposed that the mechanism for these anomalously high thermal transfer rates was the electromagnetic dipole coupling between $Ce^{3+}$ spins and $^3$He nuclear spins. On the basis of this assumption, they calculated that this so-called "magnetic Kapitza resistance" would depend linearly on $T$. Guyer[1] rederived this $T$ dependence using a different approach. Guyer went on to apply the assumption of nuclear-spin–electron-spin dipolar interaction to predict that an anomalous boundary resistance will also occur between CMN and solid $^3$He and that this resistance will be proportional to $T^2$. Leggett is in agreement with these predictions,[5] which imply that at very low $T$, solid $^3$He may be an even better medium for thermal contact to CMN than liquid $^3$He.

The method used to measure the boundary resistance was similar to that of Abel *et al.*[2] The experimental chamber which was thermally tied to a dilution refrigerator contained 0.5 g of CMN and 2.7 cm$^3$ of $^3$He. These quantities were chosen so that the specific heat of the $^3$He was always significantly greater than that of the CMN. To ensure that the CMN was coupled to the $^3$He rather than directly to the dilution refrigerator, experiments were also carried out with no $^3$He in the chamber, with only $^3$He vapor, and with liquid $^3$He. The temperature was measured by means of a resistor calibrated against the CMN susceptibility at zero magnetic field. As an added check, it was determined that at the temperatures of our measurement the relation between $\chi(0)$, the susceptibility of CMN in zero magnetic field, and $\chi(H)$, the susceptibility in a dc magnetic field $H$ parallel to the susceptibility measuring field, was given by[6]

$$\chi(0)/\chi(H) - 1 \propto H^2 \qquad (1)$$

to a high precision.

By applying a magnetic field and then reducing it to a desired field value we quickly demagnetized the CMN powder, thereby lowering its temperature ($\Delta T/T \sim 3\%$), and then observed the subsequent equilibrium time constant as it warmed back to the ambient temperature of the $^3$He. These time constants are shown in Fig. 1 at dif-