If we interchange the roles of F and k_4 and divide by 2, the second and third integrals can be expressed as an integral over the domain where the triangle inequalities are satisfied and an integral over the domain where the inequality $F+k_4>k_2$ is violated, respectively. Combining the second with the first integral we have

$$\eta_2 = -(2^3a^3/9\pi)[\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(\alpha-\tfrac{1}{2}\pi) + \int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\tfrac{1}{2}\pi)].$$

If we now permute k_2, k_4, and F so that k_4 and F successively take on the role of k_2, we may write

$$\eta_2 = -(2^3a^3/9\pi)\{\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF[\tfrac{1}{3}(\alpha+\beta+\gamma)-\tfrac{1}{2}\pi] + \int k_2\psi(k_2)\,dk_2\int k_2\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\tfrac{1}{6}\pi)\},$$

where the second integral above is taken over the domain where at least one of the triangle inequalities is violated and β and γ are the angles between k_2 and k_4 and between k_4 and F, respectively, in the triangle formed by k_2, k_4, and F. Since the sum of three angles of a triangle is π, the integrand in both integrals is the same and is equal to $-\frac{1}{6}\pi$, and the sum of the two integrals is

$$\eta_2 = -(2^3a^3/9\pi)\int_0^\infty k_2\psi(k_2)\,dk_2\int_0^\infty k_4\psi(k_4)\,dk_4\int_0^\infty F\psi(F)\,dF(-\tfrac{1}{6}\pi).$$

Taking note of Eq. (3), we see that this is equal to $\frac{1}{54}$.

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[5]We note that the second of the identities, Eq. (3), expresses the universality of the normalized coupling constant $a^{-2}U_4(0,0,0,0)$ to order ϵ. On the other hand $U_2(0)$, which determines the critical temperature, is not universal to order ϵ since B is not universal.

Evaluation of η in Wilson's Incomplete-Integration Method: Independence of Cutoff Parameters to Order ϵ^2

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The expression for the coefficient of ϵ^2 in the ϵ expansion for the critical exponent η of the Ising model has been obtained by Shukla and Green within the incomplete-integration method of Wilson. It was left by them in the form of an unevaluated integral, which appears to depend on parameters characterizing details of the renormalization-group procedure. The integral is evaluated here and found to equal $\frac{1}{54}$, regardless of the values of those parameters.

In a recent Letter,[1] Shukla and Green presented a low-order-in-ϵ solution to Wilson's incomplete-integration equation formulation[2] of the differential renormalization-group problem for the continuous version of the Ising model. The solution for the second-order-in-ϵ contribution to the critical exponent η was left in the form of an unevaluated multiple integral in which the integrand had a nontrivial dependence on a constant a and a function $\beta(k)$. If the result of integration were to depend on them, the critical exponent η would depend on the details of the renormalization-group procedure, a highly unsatisfactory result. In this note an evaluation of the integral is presented. The result is found to be $\frac{1}{54}$ independent of both a and $\beta(k)$.

The expression to be evaluated is

$$\eta = \frac{\epsilon^2 A^2}{a} \int\int \psi(|\vec{k}_1|)\psi(|\vec{k}_2|)\left[\int_0^1 \lambda \frac{d\psi(F)}{dF}\,d\lambda\right] d^4k_1\,d^4k_2, \tag{1}$$

with

$$F \equiv |\vec{k}_1 + \lambda\vec{k}_2|,$$

and

$$A^{-1} = 12\int\left[\int_0^1 \lambda\psi(\lambda|\vec{k}|)\,d\lambda\right]\psi(\vec{k})\,d^4k. \tag{2}$$

$\psi(k)$ as defined by Shukla and Green may be shown to be given by

$$\psi(|\vec{k}|) = |\vec{k}|^{-1}\,d\varphi(|\vec{k}|)/d|\vec{k}|, \tag{3}$$

with

$$\varphi(x) = x^2/2[ax^2 + c(x)], \tag{4}$$

$$c(x) = \exp\left[-\int_0^x (1/y)\beta(y)\,dy\right]. \tag{5}$$

a is a positive constant and $\beta(x)$ is a positive, monotonically increasing analytic function of x^2 with $\beta(0) = 0$. The expression for A is readily evaluated:

$$\begin{aligned} A^{-1} &= 24\pi^2 \int_0^\infty \left[\int_0^k k'\psi(k')\,dk'\right] k\psi(k)\,dk \\ &= 24\pi^2 \int_0^\infty \left[\int_0^k \frac{d\varphi(k')}{dk'}\,dk'\right]\frac{d\varphi(k)}{dk}\,dk \\ &= 24\pi^2 \int_0^\infty \varphi(k)\frac{d\varphi(k)}{dk}\,dk = 12\pi^2[\varphi(\infty)^2 - \varphi(0)^2] = \frac{\pi^2}{3a^2}. \end{aligned} \tag{6}$$

We have used the fact that $\varphi(0) = 0$ and $\varphi(\infty) = (2a)^{-1}$.

To evaluate the other integral in the expression for η we first set aside the integration over λ and write the remaining integrals in the form

$$\lambda\int x(\vec{r}_1)\exp(i\vec{k}_1\cdot\vec{r}_1)x(\vec{r}_2)\exp(i\vec{k}_2\cdot\vec{r}_2)x'(\vec{r}_3)\exp[i(\vec{k}_1 + \lambda\vec{k}_2)\cdot\vec{r}_3]\,d^4k_1\,d^4k_2\,d^4r_1\,d^4r_2\,d^4r_3, \tag{7}$$

with

$$x(\vec{r}) = (2\pi)^{-4}\int e^{-i\vec{k}\cdot\vec{r}}\psi(|\vec{k}|)\,d^4k \tag{8}$$

and

$$x'(\vec{r}) = \frac{1}{(2\pi)^4}\int e^{-i\vec{k}\cdot\vec{r}}\frac{1}{|\vec{k}|}\frac{d\psi}{d|\vec{k}|}\,d^4k. \tag{9}$$

Performing the integration over $\vec{k}_1$ and $\vec{k}_2$ in Eq. (7) we obtain

$$(2\pi)^8\lambda\int \delta^4(\vec{r}_1 - \vec{r}_3)\,\delta^4(\vec{r}_2 - \lambda\vec{r}_3)x(\vec{r}_1)x(\vec{r}_2)x'(\lambda\vec{r}_3)\,d^4r_1\,d^4r_2\,d^4r_3 = (2\pi)^8\lambda\int x(\vec{r})x(\lambda\vec{r})x'(\vec{r})\,d^4r. \tag{10}$$

Inserting our expression (3) for $\psi(r)$ we have for $x(r)$

$$\begin{aligned} x(\vec{r}) &= \frac{1}{(2\pi)^4}2\pi^2\int [J_0(kr) + J_2(kr)]k^2\frac{d\varphi}{dk}\,dk \\ &= \frac{1}{8\pi^2}\int \frac{d}{dk}\{[J_0(kr) + J_2(kr)]k^2\varphi(k)\}\,dk - \frac{1}{8\pi^2}\int 2k[J_0(kr) + J_2(kr)]\varphi(k)\,dk \\ &\qquad - \frac{1}{8\pi^2}\int k^2\varphi(k)\frac{\partial}{\partial k}[J_0(kr) + J_2(kr)]\,dk \\ &= -\frac{1}{8\pi^2}\frac{1}{r}\frac{df(r)}{dr}, \end{aligned} \tag{11}$$

where

$$f(r)=r^2\int_0^\infty[J_0(kr)+J_2(kr)]k\varphi(k)\,dk, \tag{12}$$

and $J_0(x)$ and $J_2(x)$ are zero- and first-order cylindrical Bessel functions. We have used the fact that

$$\frac{\partial y(kr)}{\partial k}=\frac{r}{k}\frac{\partial y(kr)}{\partial r}.$$

The rigorous neglect of the integral of the exact differential on the right-hand side of Eq. (11) requires the insertion of a convergence factor. We may neglect that factor for our present purposes and we will perform similar integration by parts below.

An important property of $f(r)$ is that it vanishes exponentially as $r\to\infty$ [for smooth $\beta(k)$]. We also have

$$f(0)=1/a.$$

The integration over λ is performed as follows:

$$\int_0^2\lambda x(\lambda r)\,d\lambda=\frac{1}{r^2}\int_0^r r'x(r')\,dr'=\frac{1}{8\pi^2r^2}\int_0^r\frac{df(r')}{dr'}\,dr'=-\frac{1}{8\pi^2r^2}\left[f(r)-\frac{1}{a}\right]. \tag{13}$$

We express $x'(r)$ in terms of $f(r)$ as follows:

$$\begin{aligned}x'(\vec{r})&=\frac{1}{(2\pi)^4}\int\frac{1}{|\vec{k}|^2}|\vec{k}|\frac{d\psi}{d|\vec{k}|}e^{i\vec{k}\cdot\vec{r}}\,d^4k=\frac{1}{(2\pi)^8}\int\frac{e^{i\vec{k}\cdot\vec{r}}}{|\vec{k}|^2}\left[|\vec{k}_1|\frac{d\psi}{d|\vec{k}_1|}\right]\exp[i(\vec{k}_1-\vec{k})\cdot\vec{r}_1]\,d^4r_1\,d^4k_1\,d^4k\\&=\frac{4\pi^2}{(2\pi)^8}\int\frac{1}{|\vec{r}-\vec{r}_1|^2}\left[\int|\vec{k}|\frac{d\psi}{d|\vec{k}|}\exp(i\vec{k}\cdot\vec{r}_1)\,d^4\vec{k}\right]d^4r_1.\end{aligned} \tag{14}$$

The integral involving $|\vec{k}|\,d\psi/d|\vec{k}|$ may be shown, via integrations by parts, to yield

$$\frac{2\pi^2}{r_1^3}\frac{d}{dr_1}\left\{r_1^3\frac{d}{dr_1}[f(r_1)]\right\}\equiv\frac{2\pi^2}{r_1^3}\frac{dF(r_1)}{dr_1}.$$

The angular integration over $\vec{r}_1$ yields

$$x'(r)=\frac{1}{16\pi^2}\int_0^\infty\frac{1}{r_m^2}\frac{dF(r_1)}{dr_1}\,dr_1, \tag{15}$$

where $r_m=\max\{r,r_1\}$. Using Eq. (15), Eqs. (10) and (13) yield for the integral of interest

$$\frac{\pi^4}{2}\left\{\int_0^\infty\left(\int_0^r\frac{dF}{dr'}\,dr'\right)\left(\frac{1}{ar^2}-\frac{f(r)}{r^2}\right)\frac{df}{dr}\,dr+\int_0^\infty\left[\int_0^r\left(a\frac{df}{dr'}-f(r')\frac{df}{dr'}\right)dr'\right]\frac{1}{r^2}\frac{dF}{dr}\,dr\right\}; \tag{16}$$

these final integrations are straightforward to perform. The result is

$$\frac{\pi^4}{6a^2}. \tag{17}$$

Inserting this, and our result (6) for A, into Eq. (11) yields $\eta=\epsilon^2/54$.

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