COMMENTS

Universality of the Exponent η to Order ϵ^2 for a Class of Renormalization Groups

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A formula for the exponent η to order ϵ^2 previously derived is generalized to a wider class of groups which includes the sharp-cutoff group and is shown to be universal to this order.

In a recent paper[1] the authors have determined the fixed point and the exponents for a class of renormalization groups characterized by parameters b and $\beta(k)$ (Wilson's incomplete-integration renormalization group[2] with an arbitrary cutoff function), and have shown that while to order ϵ the critical exponents are universal, in order ϵ^2 the exponent η appeared to depend on the parameters of the group. In this Letter we determine the fixed point of a somewhat more general class of renormalization groups (RNG's) which also includes the sharp-cutoff RNG as well as the linear RNG recently discussed by Bell and Wilson,[3] and show that for all members of this class the exponents are universal to order ϵ^2. The differential form of the generalized incomplete-integration renormalization-group transform is

$$\frac{\partial H}{\partial t}=\int_{\vec{k}}\left([b+\beta(k)]\sigma_{\vec{k}}+\tfrac{1}{2}d\sigma_{\vec{k}}+\vec{k}\cdot\nabla_{\vec{k}}\sigma_{\vec{k}}\right)\frac{\delta H}{\delta\sigma_{\vec{k}}}+\int_{\vec{k}}[b+\beta(k)]\gamma(k)\left(\frac{\delta H}{\delta\sigma_{\vec{k}}}\frac{\delta H}{\delta\sigma_{-\vec{k}}}+\frac{\delta^2 H}{\delta\sigma_{\vec{k}}\delta\sigma_{-\vec{k}}}\right), \tag{1}$$

where $\gamma(k)$ is an arbitrary analytic function of k^2 and the other symbols have the same meaning as in Ref. 1. If we set $\gamma(k)$ identically equal to unity, Eq. (1) reduces to Wilson's exact renormalization-group equation. The renormalization-group operator expressed by the differential equation (1) can be equivalently written as the following integral operator:

$$\exp H_t(\sigma)=\int_{\sigma'}\exp\{-\tfrac{1}{2}\int_{\vec{k}}[D(k,t)]^{-1}|\sigma_{\vec{k}}-\exp(-\tfrac{1}{2}dt)C(k,t)\sigma'_{\vec{k}e^{-t}}|^2\}\exp H_0(\sigma'), \tag{2}$$

where

$$C(k,t)=\exp[-bt-\int_{ke^{-t}}^{k}(k')^{-1}\beta(k')\,dk']$$

and

$$D(k,t)=2\int_{ke^{-t}}^{k}\frac{[b+\beta(k')]\gamma(k')}{k'}\exp\left(-2\int_{k'}^{k}\frac{b+\beta(k'')}{k''}\,dk''\right)dk'.$$

If we choose $\beta(k)=0$ for $k\leq 1$, $\beta(k)=\infty$ for $k>1$, $\gamma(k)=C$, a constant, for $k\leq 1$, and $\gamma(k)=\infty$ for $k>1$, Eq. (2) becomes equivalent to the linear RNG equation of Bell and Wilson.[3] We derive the ϵ-expansion solution for the fixed point and the critical exponents given by Eq. (1) by the same method as we used in Ref. 1. The solution for the fixed-point function $U_{20}^*(k)$ is

$$U_{20}^*(k)=\frac{k^2}{a^{-1}\exp[-2\int_0^k(k')^{-1}\beta(k')\,dk']+2\int_0^k[1+\beta(k')]\gamma(k')k'\exp[-2\int_{k'}^{k}(k'')^{-1}\beta(k'')\,dk'']}.$$

U_{21}^*, U_{41}^*, U_{42}^*, and U_{62}^* have the same expressions as those reported in Ref. 1, if one makes the following reinterpretations:

$$P(k)\to\beta(k)-2\gamma(k)[1+\beta(k)]U_{20}^*(k);\quad \psi(k)\to\gamma(k)[1+\beta(k)]\exp\int_0^k(k')^{-1}P(k')\,dk'.$$

We find that, to order ϵ, the exponents do not have any explicit dependence on a, $\beta(k)$, or $\gamma(k)$ and are the same as for the sharp-cutoff group. As before, in order ϵ^2,

$$\eta_2 = -(A^2/a)\int_{\vec{k}_2}\psi(k_2)\int_{\vec{k}_4}\psi(k_4)\int_0^1 \lambda\, d\lambda F^{-1}\, d\psi(F)/dF.$$

(The extra factor of 4 in the corresponding expression in Ref. 1 is in error.)

We shall show below that our expression for η_2 is universal. This confirms a recent numerical computation by Golner and Riedel[4] who evaluated the expression for η_2 by computer for the choices

$$\gamma(k) = 1, \quad \beta(k) = 2k^2 \text{ and } \gamma(k) = 1, \quad \beta(k) = k^4$$

and obtained $\frac{1}{54}$ in both cases.

We first notice that the function $\psi(k)$ can be written as

$$\psi(k) = -(2a)^{-1}k^{-1}d[\varphi(k)]^{-1}/dk; \quad \varphi(k) = 1 + 2a\int_0^k [1+\beta(k')]\gamma(k')k' \exp[\int_0^{k'}(k'')^{-1}\beta(k'')\,dk''].$$

This yields the important identities[5]

$$\int_0^\infty k\psi(k)\,dk = 1/2a \text{ and } A = a^2/3\pi. \tag{3}$$

Substituting the above value for A in the equation for η_2 and writing the four-dimensional integrals explicitly yields

$$\eta_2 = -(2^3a^3/9\pi)\int_0^\infty k_2\psi(k_2)\,dk_2\int_0^\infty k_4\psi(k_4)\,dk_4 I(k_2, k_4, F), \tag{4}$$

where

$$I(k_2, k_4, F) = k_2^2k_4^2\int_0^\pi \sin^2\Theta\, d\Theta\int_0^1 \lambda\, d\lambda\, F^{-1}\,d\psi(F)/dF \tag{5}$$

and Θ is the angle between vectors $\vec{k}_2$ and $\vec{k}_4$. Making use of the relation

$$d\lambda/F = dF/(k_2^2 + k_2k_4\cos\Theta),$$

we can write (5) as a Stieltjes integral,

$$I = k_2k_4^2\int_0^\pi \sin^2\Theta\, d\Theta\int_{\lambda=0}^1[\lambda/(\lambda k_2 + k_4\cos\Theta)]\, d[\psi(F) - \psi(k_4\sin\Theta)]. \tag{6}$$

Here $\psi(k_4\sin\Theta)$ has been introduced to make the integrand finite when $k_2 + k_4\cos\Theta$ vanishes. This integral may be evaluated by integration by parts and by changing the variables of integration to F and F' where $F' = k_4\sin\Theta$. The domain of integration over F and F' becomes a three-sheeted surface. The result is

$$I = \int_{k_2-k_4}^{k_2+k_4} F\psi(F)\,dF(\alpha) + \int_0^{k_4} F\psi(F)\,dF(-\pi) \text{ for } k_2 > k_4, \tag{7}$$

$$I = \int_{k_4-k_2}^{k_2+k_4} F\psi(F)\,dF(\alpha) + \int_{k_4-k_2}^{k_4} F\psi(F)\,dF(-\pi) \text{ for } k_2 < k_4, \tag{8}$$

where α is the angle between k_2 and F in a triangle whose sides are k_2, F, and k_4.

The first of the two integrals in Eqs. (7) and (8) can be combined to form an integral over the domain where k_2, k_4, and F satisfy the triangle inequalities ($k_2 + k_4 > F$; $k_2 + F > k_4$; $k_4 + F > k_2$). In Eq. (7), it is convenient to divide the domain of integration into two parts:

(i) $k_4 > F > k_2 - k_4$; (ii) $k_2 - k_4 > F > 0$.

The integral over domain (i) can be combined with the second integral in Eq. (8) to form an integral over a domain where k_2, k_4, and F satisfy the triangle inequalities as well as the inequality $k_4 > F$. The domain (ii) is the domain where $k_4 > F$ and the triangle inequality $F + k_2 > k_4$ is violated. Our expression for η_2 can be written then as the sum of three threefold integrals over k_2, k_4, and F:

$$\eta_2 = -(2^3a^3/9\pi)[\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(\alpha) + \int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\pi)$$
$$+ \int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\pi)],$$

where the domain of the first integral is over all F, k_2, and k_4 satisfying the triangle inequalities, and that of the second is the domain where in addition to the triangle inequalities the inequality $k_4 > F$ is satisfied. The third integral is over the domain where the inequality $k_4 + F > k_2$ is violated and $k_2 > k_4$ is satisfied.

If we interchange the roles of F and k_4 and divide by 2, the second and third integrals can be expressed as an integral over the domain where the triangle inequalities are satisfied and an integral over the domain where the inequality $F+k_4>k_2$ is violated, respectively. Combining the second with the first integral we have

$$\eta_2=-(2^3a^3/9\pi)[\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(\alpha-\tfrac{1}{2}\pi)$$
$$+\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\tfrac{1}{2}\pi)].$$

If we now permute k_2, k_4, and F so that k_4 and F successively take on the role of k_2, we may write

$$\eta_2=-(2^3a^3/9\pi)\{\int k_2\psi(k_2)\,dk_2\int k_4\psi(k_4)\,dk_4\int F\psi(F)\,dF[\tfrac{1}{3}(\alpha+\beta+\gamma)-\tfrac{1}{2}\pi]$$
$$+\int k_2\psi(k_2)\,dk_2\int k_2\psi(k_4)\,dk_4\int F\psi(F)\,dF(-\tfrac{1}{6}\pi)\},$$

where the second integral above is taken over the domain where at least one of the triangle inequalities is violated and β and γ are the angles between k_2 and k_4 and between k_4 and F, respectively, in the triangle formed by k_2, k_4, and F. Since the sum of three angles of a triangle is π, the integrand in both integrals is the same and is equal to $-\frac{1}{6}\pi$, and the sum of the two integrals is

$$\eta_2=-(2^3a^3/9\pi)\int_0^\infty k_2\psi(k_2)\,dk_2\int_0^\infty k_4\psi(k_4)\,dk_4\int_0^\infty F\psi(F)\,dF(-\tfrac{1}{6}\pi).$$

Taking note of Eq. (3), we see that this is equal to $\frac{1}{54}$.

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[4] G. R. Golner and E. K. Riedel, Phys. Rev. Lett. 34, 171 (1975).
[5] We note that the second of the identities, Eq. (3), expresses the universality of the normalized coupling constant $a^{-2}U_4(0,0,0,0)$ to order ϵ. On the other hand $U_2(0)$, which determines the critical temperature, is not universal to order ϵ since B is not universal.

Evaluation of η in Wilson's Incomplete-Integration Method: Independence of Cutoff Parameters to Order ϵ^2

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The expression for the coefficient of ϵ^2 in the ϵ expansion for the critical exponent η of the Ising model has been obtained by Shukla and Green within the incomplete-integration method of Wilson. It was left by them in the form of an unevaluated integral, which appears to depend on parameters characterizing details of the renormalization-group procedure. The integral is evaluated here and found to equal $\frac{1}{54}$, regardless of the values of those parameters.

In a recent Letter,[1] Shukla and Green presented a low-order-in-ϵ solution to Wilson's incomplete-integration equation formulation[2] of the differential renormalization-group problem for the continuous version of the Ising model. The solution for the second-order-in-ϵ contribution to the critical exponent η was left in the form of an unevaluated multiple integral in which the integrand had a nontrivial dependence on a constant a and a function $\beta(k)$. If the result of integration were to depend on them, the critical exponent η would depend on the details of the renormalization-group procedure, a highly unsatisfactory result. In this note an evaluation of the integral is presented. The result is found to be $\frac{1}{54}$ independent of both a and $\beta(k)$.