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Nonlinear Saturation of the Trapped-Ion Mode

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A nonlinear model of the collisional trapped-ion mode is presented, in which the energy in long-wavelength instabilities is transferred to short-wavelength modes which are then damped by ion-bounce resonances. Near marginal stability, the saturation of a single unstable Fourier mode is computed. Far from marginal stability, steady-state nonlinear solitary waves containing many Fourier modes are found. Particle transport is estimated in both cases.

It is well known[1-4] that plasma confinement in toroidal devices may be seriously impaired by the development of instabilities associated with the trapped particles, namely that class of particles which oscillate in magnetic wells created by the inherent magnetic field inhomogeneity. In the next generation of tokamaks, the ion temperature should be sufficiently large for the ions to enter the banana regime. In this parameter range (where the effective ion collision frequency $\nu_i^{\rm eff}$ is less than the trapped-ion bounce frequency ω_{bi}^T) the dissipative trapped-ion mode, a drift wave driven unstable by electron collisions, is theoretically predicted to appear.

Several authors[1-4] have studied the linear development of this instability which appears in the limit where the mode frequency ω_0 is much less than both the trapped-ion bounce frequency and the effective electron collision frequency ($\nu_- \equiv \nu_e^{\rm eff} = \nu_e/\epsilon$, where $\epsilon = r/R$ is the inverse aspect ratio). In this limit, the linear dispersion relation is

$$\omega = \omega_0 + i(\omega_0^2/\nu_- - \nu_+ - \gamma_{\rm LD}), \tag{1}$$

where $\omega_0 \simeq \epsilon^{1/2}\omega_*/2$, ω_* being the electron diamagnetic drift frequency, $\nu_+ = \nu_i^{\rm eff} = \nu_i/\epsilon$ is the effective ion collision frequency, and $\gamma_{\rm LD}$, which represents the effect of Landau damping by ion-bounce resonances,[2-4] is given by

$$\gamma_{\rm LD} = A'(1 - \tfrac{3}{2}\eta_i)\omega_0^4/(\omega_{bi}^T)^3, \tag{2}$$

where A' is a constant of order unity and $\eta_i = d\ln T_i/d\ln n$ is required to be less than $\frac{2}{3}$ to ensure Landau damping rather than growth.[2-4]

In this Letter, we study the nonlinear evolution of this mode in order to determine the saturation level of the fluctuating electric fields. Knowing the saturation level we then compute the particle transport caused by this instability. The analysis is performed using a slab model, first proposed by Kadomtsev and Pogutse,[5] which includes the nonlinear motion $\vec{E}\times\vec{B}$ of the trapped particles. Other nonlinear effects, such as particle detrapping by the electrostatic potential,[6] are not included in this treatment. The basic mechanism for the saturation of the mode is the effective transfer of energy from long-wavelength to short-wavelength modes which are then damped by ion-bounce resonances for sufficiently weak temperature gradients ($\eta_i < \frac{2}{3}$).

The basic model consists of the two-dimensional continuity equations describing the field-line-averaged $\vec{E}\times\vec{B}$ convection of the trapped particles,

$$\partial n_{e,i}^T/\partial t + c(\hat{e}\times\nabla\varphi/B)\cdot\nabla n_{e,i}^T = -\nu_{-,+}[n_{e,i}^T - \epsilon^{1/2}n_0\exp(\pm e\varphi/T)], \tag{3}$$

and the quasineutrality condition

$$n_e{}^T+(1-\epsilon^{1/2})n_0\exp(e\varphi/T)$$
$$=n_i{}^T+(1-\epsilon^{1/2})n_0\exp(-e\varphi/T). \quad (4)$$

The right-hand side of Eq. (3) describes the collisional relaxation of the trapped-particle number densities to the values that they would assume if the electrostatic potential φ were time independent. Here, n_0 is the equilibrium total number density, ϵ is the inverse aspect ratio, and hence $\epsilon^{1/2}$ is equal to the fraction of trapped particles. The effective collision frequencies $\nu_{-,+}=\nu_{e,i}/\epsilon$ are enhanced, thus accounting for the relatively small-angle scattering that is required to trap or detrap a particle.[1,5] The unit vector $\hat{e}$ is directed along the magnetic field. These equations are analyzed in the slab limit with the usual coordinates r and $r(\theta-\zeta/q)$ (ζ and θ being the toroidal and poloidal angles and q the safety factor) replaced by x and y.

In the limit $\nu_- \gg d/dt \sim \omega_0 \gg \nu_+$ appropriate to the dissipative trapped-ion mode, we combine Eqs. (3) and (4) to obtain the following nonlinear partial differential equation for the fluctuating potential $\Phi=e\varphi/T$:

$$\frac{\partial\Phi}{\partial t}+V_*\frac{\partial\Phi}{\partial y}+\frac{V_*{}^2}{\nu_-}\frac{\partial^2\Phi}{\partial y^2}-\frac{V_*}{\epsilon^{1/2}}\frac{\partial\Phi^2}{\partial y}+\nu_+\Phi=0, \quad (5)$$

where $V_*=-(\epsilon^{1/2}/2)(cT/eB)(\partial\ln n/\partial x)$ is the trapped-electron diamagnetic drift velocity. In deriving Eq. (5), we have assumed that (i) $e\varphi/T \ll 1$ so that $\exp(e\varphi/T)\simeq 1+e\varphi/T$, and that (ii) the gradient in number density is constant so that terms involving $\partial^2 n/\partial x^2$ can be dropped. We also find that the contribution of the x derivatives of φ to Eq. (5) are smaller by $\omega_0/\nu_- \equiv k_y V_*/\nu_-$ than the contributions from the y derivatives. This latter fact enables us to reduce the basic equations to the one-dimensional form.

To lowest order Eq. (5) reduces to $\partial\Phi/\partial t + V_*\partial\Phi/\partial y=0$, which describes the undamped propagation of drift waves in the direction of the electron diamagnetic drift. In the next order, moving to the drift frame $\eta=y-V_*t$ and neglecting ion collisions, one finds an equation of the form of the reversed Burgers equation. The solutions to this equation exhibit gradients that become increasingly strong (tending toward discontinuity) as $t\to\infty$. Thus, Eq. (5) demonstrates the transfer of wave energy from long to short wavelengths but it does not contain a mechanism for saturation. As noted earlier, the linear kinetic theory of this mode indicates that for sufficiently weak temperature gradients, the unstable spectrum is limited at short wavelengths by Landau damping from the ion-bounce resonances.[2-4] To incorporate this important velocity-space dissipative effect into our fluid model, we add to Eq. (5) the damping term given in Eq. (2), rewritten as $A'(1-1.5\eta_i)V_*(V_*/\omega_{bi}{}^T)^3\partial^4\Phi/\partial\eta^4$. When linearized, Eq. (5) now correctly reproduces the dispersion relation given by Eq. (1). The Landau damping by ion-bounce resonances is the mechanism through which the plasma absorbs the energy transferred from long to short wavelengths by the nonlinearity, and thus quenches the instability after a finite time of growth. Introducing the dimensionless variables $\tau=(\omega_0{}^2/\nu_-)t$, $\xi=\eta/r$, and $\psi=(\nu_-/\epsilon^{1/2}\omega_0)\Phi$, where $\omega_0=V_*/r$, the equation describing the growth and saturation of the fluctuating potential becomes

$$\frac{\partial\psi}{\partial\tau}+\frac{\partial^2\psi}{\partial\xi^2}+\alpha\frac{\partial^4\psi}{\partial\xi^4}+\nu\psi+\frac{\partial\psi^2}{\partial\xi}=0, \quad (6)$$

where $\alpha=A'(1-1.5\eta_i)(\omega_0/\omega_{bi}{}^T)^2\nu_-/\omega_{bi}{}^T$ is a measure of the relative strength of Landau damping compared to the electron collisional growth, and $\nu=\nu_-\nu_+/\omega_0{}^2$. Solutions to Eq. (6) are required to satisfy periodic boundary conditions $\psi(\xi)=\psi(\xi+2\pi)$ in accordance with toroidal periodicity, over a length $2\pi r$.

In this Letter, approximate solutions are obtained for conditions near marginal stability, and for unstable situations in which many modes are excited. The distinction between the two cases is determined by the magnitude of the Landau damping factor α.

The Fourier representation

$$\psi(\xi,\tau)=\sum_1^\infty\psi_n(\tau)\sin n\xi \quad (7)$$

satisfies the periodic boundary conditions and has the odd parity, $\psi(\xi)=-\psi(-\xi)$, demanded by Eq. (6). Substituting this into Eq. (6) gives

$$\partial\psi_n/\partial\tau-\gamma_n\psi_n=\tfrac{1}{2}n\textstyle\sum_m(\psi_{n-m}\psi_m-\psi_m\psi_{m+n}), \quad (8)$$

where $\gamma_n=n^2-\alpha n^4-\nu$ is the linear growth rate of the nth Fourier mode. For negligible ion collisional damping ($\nu\to 0$), we note that if $0.25<\alpha<1$ then only the $n=1$ mode is unstable, while if $\alpha\ll 1$ then all modes with $n<\alpha^{-1/2}$ will be unstable. Near marginal stability where there is only a single unstable mode with a small growth rate, Eq. (8) can be solved by a mode-coupling calculation. Letting $n=p$ be the single unstable mode, it is seen that the mode grows till the quadratic nonlinearity effectively couples it to the nonlinearly generated damped mode $n=2p$. The equations describing coupling between the two modes are

$$\partial\psi_p/\partial\tau-\gamma_p\psi_p=-p\psi_p\psi_{2p},\quad \gamma_p>0, \tag{9}$$

$$\partial\psi_{2p}/\partial\tau-\gamma_{2p}\psi_{2p}=p\psi_p^{\,2},\quad \gamma_{2p}<0, \tag{10}$$

which can be solved to obtain expressions for $\psi_p(\tau)$ and $\psi_{2p}(\tau)$. We note that for $\tau\gg\gamma_p^{-1}$, $\psi_p\to(\gamma_p|\gamma_{2p}|)^{1/2}/p$ and $\psi_{2p}\to\gamma_p/p$.

The case $0<\alpha\ll1$, far from marginal stability, can be studied by using a multiple scale-length expansion to obtain a steady-state solution to Eq. (6). In a time comparable to the fastest growing linear mode the instability will saturate, thus solutions to the steady-state equation will be approached asymptotically for $\tau\gg\gamma_{\max}^{-1}\sim4\alpha$. With $\nu=0$, the equation to be solved is

$$\frac{\partial^2\psi}{\partial\xi^2}+\alpha\frac{\partial^4\psi}{\partial\xi^4}+\frac{\partial\psi^2}{\partial\xi}=0, \tag{11}$$

where

$$\psi(\xi+2\pi)=\psi(\xi)=-\psi(-\xi).$$

If the nonlinear term in Eq. (11) were missing, then the periodic solutions of the linear equation would have a spatial dependence like $\sin\xi/\alpha^{1/2}$. Since $\alpha\ll1$, this space scale variation is much faster than ξ. Guided by this, we introduce two space scales and look for solutions which may be expanded in the form

$$\psi=A(\xi)+C_1(\xi)\sin[\xi/\alpha^{1/2}+\alpha^{1/2}\textstyle\int^\xi\mu_1(\xi)d\xi]+\alpha^{1/2}C_2(\xi)\sin[2\xi/\alpha^{1/2}+\alpha^{1/2}\int^\xi\mu_2(\xi)d\xi]+\dots, \tag{12}$$

where the quantities $A(\xi)$, $C_1(\xi)$, etc., are also to be expanded in power series in $\alpha^{1/2}$. We substitute Eq. (12) into Eq. (11) and keep terms of order $\alpha^{-1/2}$ and terms of order unity. After some straightforward algebra, we find

$$\partial C_1/\partial\xi-AC_1=0,\quad \partial^2A/\partial\xi^2+[\partial(A^2+\tfrac12C_1^{\,2})/\partial\xi]=0, \tag{13}$$

$$2C_1\mu_1=3\partial^2C_1/\partial\xi^2-C_1^3/12,\quad C_2=-C_1^2/12. \tag{14}$$

Equations (13) are solved simultaneously to obtain

$$C_1=C_m\,\mathrm{nd}(\gamma\xi,k), \tag{15}$$

$$A=\gamma k^2\,\mathrm{sn}(\gamma\xi,k)\,\mathrm{cn}(\gamma\xi,k)\,\mathrm{nd}(\gamma\xi,k), \tag{16}$$

with $\lambda=C_m/2k'$ and $k'^2=1-k^2$, where λ and k are unknown constants. The Jacobi elliptic functions $\mathrm{nd}(u,k)$, $\mathrm{sn}(u,k)$, and $\mathrm{cn}(u,k)$ are periodic functions of u with a periodicity $2K(k)$ for both nd and the product of sn with cn, $K(k)$ being the complete elliptic integral of the first kind.

From the periodic boundary conditions, $\psi(\xi+2\pi)=\psi(\xi)$, it follows that $A(\xi+2\pi)=A(\xi)$ and $C_1(\xi+2\pi)=C_1(\xi)$. The periodicity of A and C_1 is assured by the choice $2\pi\lambda=2K(k)$, whence $\pi C_m=2k'K(k)$. The periodicity on the slow scale of the sine function in Eq. (12) demands that

$$\frac{l}{\alpha^{1/2}}-\frac{1}{\alpha}=\frac{K(k)}{2\pi^2}[3(2-k^2)K(k)-\tfrac{19}{3}E(k)]\equiv F(k), \tag{17}$$

where l is the integer nearest to $1/\alpha^{1/2}$, and $E(k)$ is the complete elliptic integral of the second kind. The function $F(k)$ varies between $-\frac{1}{24}$ for $k=0$ and ∞ for $k=1$. The solution is then

$$\psi=\frac{K}{\pi}\,\mathrm{nd}\left(\frac{K\xi}{\pi},k\right)\left[k^2\,\mathrm{sn}\left(\frac{K\xi}{\pi},k\right)\mathrm{cn}\left(\frac{K\xi}{\pi},k\right)+2(1-k^2)^{1/2}\sin\left(\frac{\xi}{\alpha^{1/2}}\right)\right] \tag{18}$$

valid in the limit $\alpha\ll1$. The form of ψ for $k\simeq1$ is shown in Fig. 1. For moderate values of k, $K(k)\sim\frac12\pi$, hence $\psi\sim\frac12$ and $e\varphi/T\sim\epsilon^{1/2}\omega_0/\nu_-$. The detrapping potential is $e\varphi/T\sim(B_{\max}-B_{\min})/B_{\max}\sim\epsilon$; thus, for $\omega_0/\nu_-<\epsilon^{1/2}$, we expect detrapping to be less important than the mechanism reported here.

With the fluctuating electrostatic potentials driven by the instability calculated in the two limits $\alpha\sim1$ and $\alpha\ll1$, we can compute the particle transport by use of the expression

$$D=\frac{\epsilon^{1/2}}{\nu_-}\left(\frac{cT}{eB}\right)^2\left\langle\left(\frac{\partial\Phi}{\partial y}\right)^2\right\rangle=\frac{\epsilon^{3/2}}{\nu_-}\left(\frac{\omega_0}{\nu_-}\right)^2\left(\frac{cT}{erB}\right)^2\left\langle\left(\frac{\partial\psi}{\partial\xi}\right)^2\right\rangle \tag{19}$$

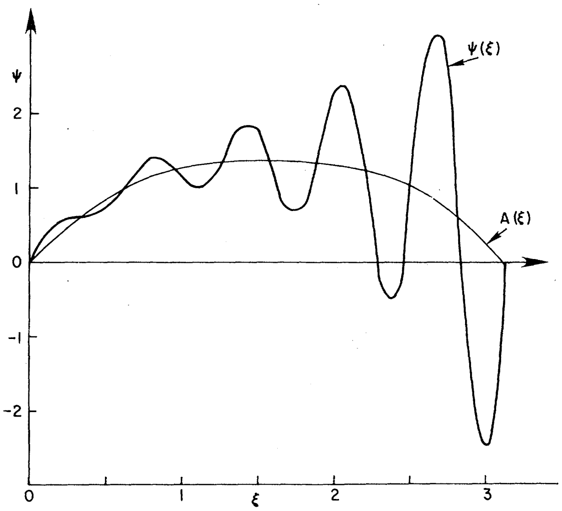

FIG. 1. Typical potential wave form computed using Eq. (18) with $\alpha = 0.01$ and $k \simeq 1$. Note that $\psi = (\nu_- / \epsilon^{1/2}\omega_0)(e\varphi/T)$.

for the diffusion coefficients.

$$D = \frac{\epsilon^{3/2}}{\nu_-}\left(\frac{\omega_0}{\nu_-}\right)^2\left(\frac{cT}{erB}\right)^2 \gamma_p |\gamma_{2p}| \quad \text{for } \alpha \sim 1, \qquad (20)$$

$$D = \frac{\epsilon^{3/2}}{\nu_-}\left(\frac{\omega_{bi}{}^T}{\nu_-}\right)^3\left(\frac{cT}{erB}\right)^2 f(k) \quad \text{for } \alpha \ll 1, \qquad (21)$$

where $f(k) = 2E(k)K(k)/\pi^2$; over a broad range of k within $0<k<1$, we have $f(k) \sim 0.5-1.0$.

In this paper we have analyzed the nonlinear evolution of the dissipative trapped-ion mode for conditions near marginal stability ($\alpha \sim 1$) and for situations where many linearly unstable modes are present ($\alpha \ll 1$). In each case the convective nonlinearity transfers energy from long-wavelength modes to short-wavelength modes which are then Landau damped by ion-bounce resonances. The saturated potentials have been computed and used to calculate the diffusion that results from the instability. The diffusion coefficient is a strongly increasing function of temperature. However, for high temperatures the ratio of the mode frequency to the effective electron collision frequency, ω_0/ν_-, which is an expansion parameter in both the usual linear theory and our nonlinear analysis, becomes of the order unity or more. Hence, in this regime our analysis ceases to be valid. Parameters of next-generation tokamaks typically fall into the regime $\omega_0/\nu_- \ll 1$, and so it is of interest to compare the relative magnitudes of the diffusion coefficient given in Eq. (21) to one suggested by Kadomtsev and Pogutse,[5] namely $D_{KP} \sim (r/R)^{5/2}(cT_e/eB)^2/4\nu_e r_n{}^2$. For parameters characteristic of the PLT tokomak ($n \sim 5\times10^{13}/\text{cm}^3$, $R = 130$ cm, $r_n = 45$ cm, $B = 50$ kG, $q = 2.5$), we find that the ratio of our diffusion coefficient to that given by Kadomtsev and Pogutse is $D/D_{KP} = 3\times10^{-2}T^6$ where T is in keV, while the basic perturbation parameter of the linear theory $\omega_0/\nu_- = 1.0\times10^{-3}T^{5/2}$. Thus, for temperatures less than approximately 2 keV we find that diffusion caused by the trapped-ion mode in the PLT tokomak will be less than that predicted by Kadomtsev and Pogutse.

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