from the experimental line scans and the uncertainty in its magnitude led to the uncertainties in ΔE_P shown in Table I.

The contribution to the uncertainty of the rf power calibration was estimated to be ± 0.02 MHz. Other possible sources of error which were found to be negligible (<0.01 MHz) were Doppler shift, time dilation, rf Stark shifts, stray electric fields, and residual magnetic fields.

Table I summarizes the results of each run. The weighted average result $\Delta E_P = 59.22(14)$ MHz is in good agreement with the theoretical prediction of 59.1501(1) MHz.[9] The experimental precision indicates that techniques of subnatural-linewidth spectroscopy can be successfully used for the measurement of unresolved features of atomic structure.

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Electron Density at the Nucleus and the Lamb Shift of ^3P Levels of Two-Electron Ions*

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The Lamb shifts of the $2\,^3S$ and $2\,^3P_1$ levels in He I and Li II have been calculated by a method of effective-oscillator-strength sums. The computed shifts are in a good agreement with experiment. For Li II, the theoretical Lamb shift of the $2\,^3P_1$ level is $+0.291 \pm 0.041$ cm^{-1} to be compared with the experimental value of $+0.230 \pm 0.060$ cm^{-1} derived from the $2\,^3S$-$2\,^3P_1$ interval measured by Bacis and Berry. A significant Lamb shift of ^3P levels is attributed to an anomalously low electron density at the nucleus in these states.

Recently the author gave[1] a theoretical discussion of the Lamb shift in excited states of two-electron ions. Numerical calculations performed for the ions of nuclear charge $Z \geq 4$ predicted a significant Lamb shift of ^3P atomic levels. Independently, Berry and Bacis[2] reported hfs measurements which could suggest a similar effect in the singly ionized lithium atom. Since that time, high-resolution studies of the $2\,^3S$-$2\,^3P$ transition have been accomplished[3] for ^{6,7}Li II. The experiment shows a difference,

$$D_L^{\text{expt}} = -1.2547 \pm 0.0040 \text{ cm}^{-1}, \quad (1)$$

between the observed value of the $2\,^3S$-$2\,^3P_1$ interval in ^{7}Li II and that calculated by Accad, Pekeris, and Schiff[4] to order α^2 and corrected with respect to the singlet-triplet mixing of the P states[5] as well as with respect to hf interactions.[3] Berry and Bacis estimate the Lamb shift of the $2\,^3S$ level to be -0.99 ± 0.04 cm^{-1}. This leaves the difference

$$D_L' = +0.2647 \pm 0.04 \text{ cm}^{-1} \quad (2)$$

to be accounted for.

The difference D_L' is appreciably bigger than an expected Lamb shift of ± 0.007 cm^{-1} for a $2p$ electron in the hydrogenic ion of $Z = 2$. Therefore the suggestion that this difference can still be attributed to the Lamb shift of the $2\,^3P_1$ level needs an independent justification. Berry and Bacis pointed out[2] that an underestimation of the S shift and a slow convergence of the method of

Accad, Pekeris, and Schiff[4] for nonrelativistic S states could be other possible explanations of the significant difference D_L'.

In this communication we shall discuss the effect and we shall find that D_L' is, in fact, in good agreement with the theoretical value of the $2\,^3P_1$ Lamb shift obtained here.

Electron density term.—Following the notations used by the author earlier,[1] we shall write the Lamb shift E_I of the relativistic potential of ionization of a two-electron ion in a $|nLSJ\rangle$ state as

$$E_{I,\alpha^3} = \delta E(nLS) - E_{L,2}''(nLS) - E_{L,2}'''(nLSJ), \quad (3)$$

where all corrections of orders $Z^4\alpha^3$, $Z^3\alpha^3$, and $Z^3\alpha^3 \ln\alpha$ have been included in (3) (energies are in rydbergs). The main contribution to the shift E_I comes from $\delta E = \delta E_1 + \delta E_2$:

$$\delta E_1 = -(8\alpha^3 Z^4/3\pi)[2\ln(1/\alpha) + \tfrac{19}{30}]\Delta(Z) \quad (4)$$

and

$$\delta E_2 = -(\alpha^3/2\pi)[P(Q) - P(q)], \quad (5)$$

where

$$P(Q) = \textstyle\sum_{Q'} F_{Q'Q}(E_{Q'} - E_Q)^2 \ln|(E_{Q'} - E_Q)| \quad (6)$$

is related to the hydrogenic ion with nuclear charge Z, $F_{Q'Q}$ being oscillator strengths for electric dipole transitions from the initial state $|Q\rangle = |1s\rangle$ to all other states $|Q'\rangle$ of the ion. The second term in (5) has an analogous meaning for the two-electron ion of the same Z, $|q\rangle$ being the initial state $|nLS\rangle$.

The electron density at the nucleus in the initial state $|q\rangle$ of the two-electron ion is determined by $\langle q|\delta^{(3)}(\vec{r}_1)|q\rangle$. This quantity enters the term δE_1, formula (4), since the correlation function $\Delta(Z)$ is given by

$$\Delta(Z) = 2\pi\langle q|\delta^{(3)}(\vec{r}_1)|q\rangle/Z^3 - 1. \quad (7)$$

The functions $\Delta(Z)$ computed from the data of Ref. 4 for $n\,^{1,3}S$ and $n\,^{1,3}P$ states, $n \leq 5$, of the ions with $Z \leq 10$ are presented in Fig. 1, scaled by a factor of n^3. It can be asserted that, for low Z, these functions differ considerably from the asymptotic values $\Delta(\infty)$ for all but $n\,^1P$ states.

In the case of $n\,^{1,3}S$ and $n\,^1P$ states, $\Delta(Z) > 0$, which corresponds to a regular situation when the electron density at the nucleus in a $|q\rangle$ state of the two-electron ion is higher than that in the $|1s\rangle$ state of the hydrogenic ion. For a positive $\Delta(Z)$, the term δE_1, formula (4), is negative and the resulting shift decreases the potential of ionization. For S states, the departure of $\Delta(Z)$ from

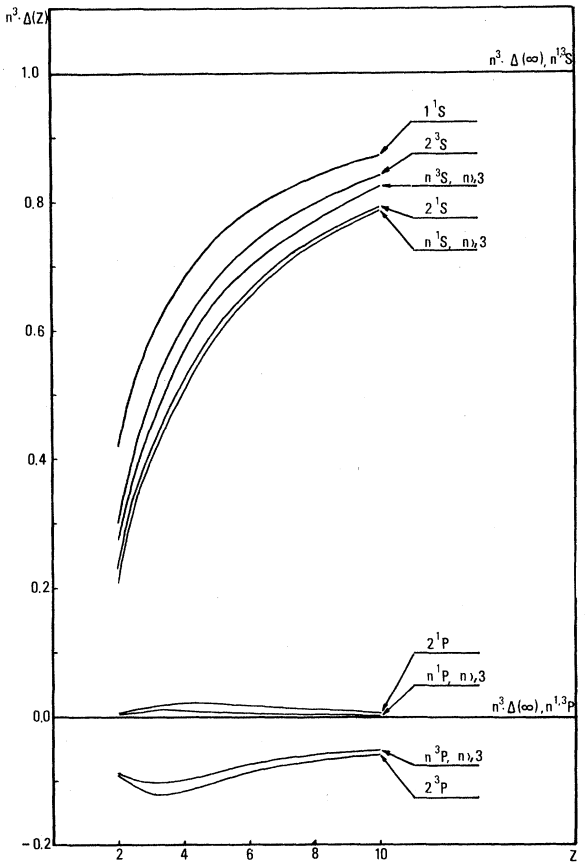

FIG. 1. The correlation functions $n^3\Delta(Z)$ for $n\,^{1,3}S$ and $n\,^{1,3}P$ states of two-electron ions.

$\Delta(\infty)$ tends to diminish this effect.

The ^3P states are a rather special case: $\Delta(Z) < 0$; that is, the electron density at the nucleus is *lower* in these states than that in the $|1s\rangle$ state of the hydrogenic ion. The function $\Delta(Z)$ attains a minimum at $Z \approx 3.5$ where its magnitude is about 25% of that for S states, instead of approaching the asymptotic value $\Delta(\infty) = 0$. This leads to a significant shift δE_1 which increases the potential of ionization. We can expect that the behavior of functions $\Delta(Z)$ of $L > 1$ states will be similar to that of the P states.

For both S and P states of low Z, the electron-density term δE_1 gives the main contribution to the total Lamb shift E_I. The term δE_2, depending upon the oscillator-strength sums, constitutes about 30% of δE_1.

Effective-oscillator-strength sums.—For calculation of two-electron oscillator sums (6) we shall use the following method. Let us assume that the initial state $|q\rangle$ is given by

$$|q\rangle = 2^{-1/2}\mathcal{A}(1,2)u_{1s}(1,Z_i)u_{nL}(2,Z_a) \quad (8)$$

and that

$$|q'\rangle = 2^{-1/2}\mathcal{A}(1,2)\hat{C}(\beta,\gamma|q)u_\beta(1,Z_i)u_\gamma(2,Z_a), \tag{9}$$

where $u_\beta(1,Z_i)$ is the hydrogenic function of electron 1 in a $|\beta\rangle$ state for the Coulomb field of charge Z_i, and $\hat{C}$ is a coupling operator for transitions from $|q\rangle$ to $|q'\rangle$. The term δE_2 given by (5) and (6) then becomes, for a fixed value of Z, a two-parameter function of Z_i and Z_a:

$$\delta E_2(Z_i,Z_a) = -(8\alpha^3Z^4/3\pi) \times\{\ln[K_0(1s)Z^2] - (Z_i/Z)^4\ln[K_0(1s)Z_i^2] - (1/n^3)(Z_a/Z)^4\ln[K_0(nL)Z_a^2] \pm A_L(Z_i,Z_a)\}, \tag{10}$$

$K_0(nl)$ being the Bethe average excitation energy of an electron in an $|nl\rangle$ state with $Z=1$.

The last term in (10), that is $A_L(Z_i,Z_a)$, accounts for electron exchange. This term vanishes when $Z_i = Z_a$.

We specify the numerical values of Z_i and Z_a by the condition

$$S_k(Z_i,Z_a) = S_k(Z), \quad k=1,2. \tag{11}$$

In this equation, $S_k(Z)$ is the oscillator-strength sum $\sum_q f_{q'q}(\mathcal{E}_{q'}-\mathcal{E}_q)^k$ for the $|q\rangle$ state of the two-electron ion. With $|q\rangle$ being the *exact* wave function of the state, the sums $S_k(Z)$, $k=1,2$, are equivalent to the expectation values given by Eqs. (12) and (13) below, provided energies $\mathcal{E}_{q'}$ are expressed in rydbergs:

$$S_1(Z) = \tfrac{4}{3}\langle q|(\vec{p}_1+\vec{p}_2)^2|q\rangle, \tag{12}$$

$$S_2(Z) = (16\pi Z/3)\langle q|\delta^{(3)}(\vec{r}_1)+\delta^{(3)}(\vec{r}_2)|q\rangle. \tag{13}$$

The left-hand side of Eq. (11) denotes the expectation values (12) and (13), obtained with wave functions (8) and (9). The values of parameters Z_i and Z_a obtained from (11) are then used to compute the effective sum (10) which gives the contribution to the Lamb shift.

Lamb shifts in He and Li$^+$.—I have computed Lamb shifts for $n=2$, ^3S and ^3P states of He I and Li II. The numerical values of $S_k(Z)$, required for calculation of Z_i and Z_a, have been obtained from the expectation values of Ref. 4.

TABLE I. The S states of He I. Comparison of different computational methods for the shift $\delta E = \delta E_1 + \delta E_2$ (in cm^{-1}).

State	Present method with $A(Z_i,Z_a)=0$	Direct summation over $\|q'\rangle$
$1\,^1S$	−1.497	−1.475 ± 0.021[a]
$2\,^1S$	−0.103	−0.119 ± 0.016[b]
$2\,^3S$	−0.133	−0.129 ± 0.011[b]

[a]Ref. 7. [b]Ref. 8.

For neutral helium, the present method can be compared with previous calculations[7,8] of Lamb shifts in S states. Table I gives values of δE for $1\,^1S$ and $2\,^{1,3}S$ states of He, which have been obtained by the present method with the exchange terms $A(Z_i,Z_a)$ being neglected in (10), (12), and (13). Comparison with calculations by Salpeter and Zaidi[7] and by Suh and Zaidi[8] shows that for low-lying S states the present method without exchange terms leads to δE which coincide, within the stated probable error, with the values of δE obtained by direct summation of the oscillator-strength terms.

In view of this, I have also neglected the exchange terms $A(Z_i,Z_a)$ in my calculations of Lamb shifts for ^3P states.[9]

The computed Lamb shifts to order α^3 are presented in Table II. The total shift E_I also includes

TABLE II. The Lamb shifts in triplet states of He I and Li II, $E_I = \delta E_1 + \delta E_2 - E_{L,2}'' - E_{L,2}''' - E_{L,2}''''$. All shifts are given in inverse centimeters.

	He I		Li II	
Term	$2\,^3S$	$2\,^3P_1$	$2\,^3S$	$2\,^3P_1$
	$Z_i = 2.012$, $Z_a = 1.126$	$Z_i = 1.994$, $Z_a = 0.972$	$Z_i = 3.023$, $Z_a = 2.114$	$Z_i = 2.989$, $Z_a = 1.954$
δE_1	-0.224	0.070	-1.905	0.444
δE_2	0.091	-0.032	0.910	-0.155
$-E_{L,2}'' - E_{L,2}'''$	0.000	-0.001	0.000	-0.006
$-E_{L,2}''''$	-0.002	0.001	-0.030	0.008
E_I	-0.135	+0.038	-1.025	+0.291
E_I(S-P)	-0.173		-1.316	
Experiment	-0.175 ± 0.01[a]		-1.255 ± 0.004[b]	

[a]Ref. 10. [b]Ref. 3.

the leading term

$$-E_{L,2}''' = -8\alpha^4 Z^5(1+\tfrac{11}{128}-\ln 2/2+\tfrac{5}{192})\Delta(Z) \quad (14)$$

of order α^4 which is a generalization of the term[10] for relativistic corrections to the Lamb shift in the one-electron ion to the case of a two-electron ion.

Comparison with experimental results for the $2\,^3S$-$2\,^3P_1$ transition in He I[11] and in Li II[3] shows that inclusion of the Lamb shift of the P level is essential to bring the experimental and theoretical total shifts $E_I(S\text{-}P)$ in agreement with each other. The J-dependent term $-E_{L,2}'''$ contributes only 2% to the shift $E_I(2\,^3P_1)$ so that the latter is determined, for low Z, mainly by correlation effects in δE_1.

For Li II the value $E_{I,\alpha^3}(2\,^3S) = -0.995$ cm^{-1} confirms the estimate[2] by Berry and Bacis. The main uncertainty in the shift $E_I(2\,^3S)$ is due to an error η_1 in δE_2. We shall adopt $\eta_1(2\,^3S) = \pm -0.040$ cm^{-1} stated in Ref. 2. The uncertainty $\eta_2(2\,^3S)$, due to terms of orders α^4 and $\alpha^3(m/M)$ neglected in (14), is probably ± 0.015 cm^{-1}. The volume-isotope correction contributes about 0.0008 cm^{-1}.

For the $2\,^3P_1$ state of Li II, no other calculations of the oscillator-strength sums δE_2 are available. We shall obtain a crude estimate of the error η_1 by assuming that the calculation of δE_2 doubles the uncertainty brought in the term δE_2 of $2\,^{1,3}S$ He by the method of Suh and Zaidi.[8] This gives $\eta_1(2\,^3P_1) = \pm 0.037$ cm^{-1}. The probable error $\eta_2(2\,^3P_1)$ is ± 0.004 cm^{-1}, and the volume-isotope correction may definitely be neglected.

Within these estimates of uncertainties in the theoretical shifts, one obtains for ^{7}Li II

$$E_I(2\,^3S) = -1.025 \pm 0.055 \text{ cm}^{-1} \quad (15)$$

and

$$E_I(2\,^3P_1) = +0.291 \pm 0.041 \text{ cm}^{-1}. \quad (16)$$

The experimental value of the Lamb shift of the $2\,^3P_1$ level is then found thus:

$$E_I^{\text{expt}}(2\,^3P_1) = E_I(2\,^3S) - D_L^{\text{expt}} = +0.230 \pm 0.059 + \eta_3 \text{ cm}^{-1}, \quad (17)$$

where D_L^{expt} is given by (1) and η_3 accounts for the uncertainty in the nonrelativistic position of the two levels, mainly of the $2\,^3S$ level. Despite slow convergence of the Pekeris method for low-lying S states,[12] it appears that the Pekeris estimate $\eta_3 = \pm 0.001$ cm^{-1} for $2\,^3S$ Li II is reliable. This brings us to

$$E_I^{\text{expt}}(2\,^3P_1) = +0.230 \text{ cm} \pm 0.060 \text{ cm}^{-1} \quad (18)$$

which is in good agreement with the theoretical value (16). I shall emphasize that the main error in the present experimental value (18) is due to the uncertainty in the theoretical value of the Lamb shift of the $2\,^3S$ level.

For ^1P states, the terms δE_1 and δE_2 are of comparable magnitude and opposite signs so that they nearly cancel each other. This results in a *small* total shift of the ^1P level. Recent experiments[13] for two-electron ions of low Z support this conclusion.

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