Estimate of the $(Z\alpha)^2\alpha^2$ Vacuum Polarization Term in Muonic Pb†

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We have calculated the static monopole contributions to the $(Z\alpha)^2\alpha^2$ vacuum polarization correction for the $4f_{5/2}$ state in muonic Pb. We find an energy shift of -1 ± 0.1 eV (increased binding). We argue that neglected higher multipole contributions have the same sign and similarly small magnitude, thus increasing by a slight amount the current discrepancy between muonic atom theory and experiment.

There exists a persistent discrepancy between theory and experiment in the high μ-atomic states of heavy nuclei. In the case of Pb, for example, the measured $5g_{7/2}-4f_{5/2}$ transition energy is greater than the theoretical prediction by 62 ± 21 eV out of a transition energy of 437 687 eV. Thorough reviews of the theoretical calculations exist in the literature.[1]

Recently, it has been suggested[2] that the vacuum polarization terms of order $(Z\alpha)^2\alpha^2$, depicted in Fig. 1(a), may contribute significantly. Since the experiment provides an important test of quantum electrodynamics, it is important to have reliable calculations or estimates of any contributory effects. We[3] (and others[4]) have previously calculated all diagrams of order $(Z\alpha)^n\alpha$, n odd [Fig. 1(b)], for nuclei of finite size. The same program has been employed here to estimate the contribution of Fig. 1(a). This is possible because the muon velocity is sufficiently slow ($v^2/c^2\sim0.02$ in $4f$ Pb) that transverse components of the electromagnetic field can be neglected. Thus the effect can be represented in second-order perturbation theory by using the static Coulomb interaction,

$$\Delta E=\sum_{pq\mu}\frac{|\langle pq\mu|V_{e\mu}|\mu_0\rangle|^2}{E_{\mu_0}-E_\mu-\epsilon_p-\epsilon_q},\tag{1}$$

where μ is the set of muonic quantum numbers, μ_0 is the initial muonic state, p (q) are electron (positron) quantum numbers, and ϵ_p (ϵ_q) > 0 is the electron (positron) excitation energy in the field of the nucleus. We did not use Eq. (1) explicitly, but it serves to clarify our calculation.

We distinguish two approximations which bound the second-order perturbation:

(I) The intermediate muon states (μ) are limited to the single initial state μ_0. In this case the muon appears as a static charge with the initial state distribution. This probably underestimates the shift since all higher intermediate states enter with the same (negative) sign, and our results indicate that the lower-lying muon states contribute negligibly.

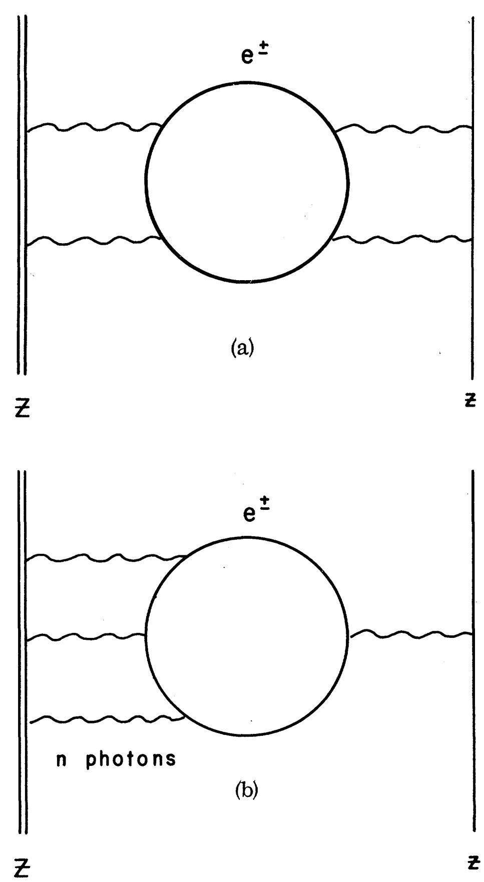

FIG. 1. Vacuum-polarization contributions to muonic-atom energy levels.

(II) The muon excitation energy is neglected. This permits closure over the muon intermediate states. One calculates the energy for a point muon as a function of position of the muon and then averages over the initial muon density distribution. This gives an upper bound to the shift subject to the same qualification mentioned in (I).

In either case, one calculates the energy to second order in the charge (say z) of the muon. The muon polarizes the vacuum linearly (order z) and then interacts (order z) with the polarization field. It follows from elementary considerations that since we are considering the linear response of the field, the interaction energy is

$$\Delta E = \tfrac{1}{2}\int \varphi_{\mathrm{vp}}(\vec{r})\rho_\mu(\vec{r})\, d^3r, \tag{2}$$

where φ_{vp} is the induced potential and $\rho_\mu = -e|\psi_\mu|^2$.

As reported in Ref. 3, we calculate the Dirac wave functions and resulting vp electron-positron density in the given "external" electric field, here taken to be due to both the nucleus and the muon. Numerical subtraction of the vp density linear in the "external" potential removes terms of order $Z\alpha$ and $z\alpha$. The result at this point is finite and requires no renormalization (see, however, Ref. 3). Further subtraction of the vp density due to the nucleus alone and to the muon alone removes terms of order $(Z\alpha)^0(z\alpha)^m$ and $(Z\alpha)^n(z\alpha)^0$. The remaining density is of order $(Z\alpha)^n(z\alpha)^m$, $n \geq 1$, $m \geq 1$, $n+m \geq 3$ and odd. Even in the usual point perturbation theory, all contributions to these remaining orders are finite and thus contain no ambiguities associated with infinite subtractions. The energy shift in any order m is given by Eq. (2) with the factor 2^{-1} replaced by $(m+1)^{-1}$.

Consider first restricting the calculation to good $\kappa = -1$ (or $+1$), where κ is the Dirac quantum number for the electron-positron states. Then the "external" muon charge is automatically averaged over solid angle. The result of this calculation [approximation I, $\kappa = \pm 1$] is $\Delta E(4f_{5/2}) = -1$ eV. In this case, the muon charge is a diffuse shell of thickness ~50 fm with $\langle 1/r\rangle = 1/50$ fm. The same calculation using approximation II (an infinitesimal shell at 50 fm) produced the same result within the limits of our expected numerical accuracy[5] (<0.1 eV). Thus the effect of radial localization is small, i.e., excitation of the muon to other than the initial state is unimportant. For these calculations, we scaled the "external" muon charge to $z = -5$ and then rescaled the result in order to ensure sufficient accuracy. A repeat of the calculation for $z = -2.5$ verified the linearity of the effect and thus the smallness of the terms which are higher order in z.

The contribution of $|\kappa| = 2$ electron-positron states was also calculated for a spherically averaged muonic charge. The added contribution had the same sign and was even smaller, about 10% of the $|\kappa| = 1$ contribution. We do not regard this result as definitive because of the previously mentioned numerical limit of accuracy. Although it is possible, a more accurate calculation does not appear to be warranted.

A proper calculation for approximation II requires solving the two-center (nucleus and muon) electron problem. This would result in admixing of various κ states through higher multipole interactions. Such a calculation is not practicable with our program. However, the smallness of the effect due to the radial localization and to the spherical $|\kappa| = 2$ contribution is convincing evidence that angular localization is also a small effect. The $|\kappa| = 1$ to $|\kappa| = 2$ off-diagonal mixing contribution should be down from the $|\kappa| = 1$ shift by the order of the square root of the $|\kappa| = 2$ reduction. A generous estimate of all contributions for $|\kappa| > 1$ would be to set them equal to the $|\kappa| = 1$ shift. We also see no reason to expect that the dipole contributions from $\kappa = +1$ to $\kappa = -1$ mixing are significantly greater than the $|\kappa| = 1$ monopole contributions. Thus we believe that the neglected higher multipole interactions add less than 2 eV to the 1 eV calculated for the static monopole interaction.

We conclude that the contribution of the vacuum polarization diagram $(Z\alpha)^2\alpha^2$ [Fig. 1(a)] is less (and perhaps considerably less) than 3 eV for the $5g \to 4f$ transitions in Pb, and in the direction to *increase* the reported discrepancy.[1] This is consistent with the interpretation that the effect [which is roughly $-1/Z$ of the $(Z\alpha)^3\alpha$ term] can be attributed to shielding of the nucleus by the muonic charge, so that the insertion of the muon as a source into the higher-order graphs weakens their total effect.

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[5]This estimate is obtained by noting that charge is conserved in the induced density to about 5%, i.e., $[\int_0^\infty \rho_{\rm vp}(r) r^2\, dr]/[\int_0^\infty |\rho_{\rm vp}(r)| r^2\, dr] \approx 0.05$. We have conservatively increased this fraction by a factor of 2 when estimating the accuracy of the energy level shift.

Radiative Corrections of Order α^2 in Muonic Atoms of Heavy Nuclei*

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A part of the hitherto neglected radiative corrections of order α^2 is calculated. Its inclusion in the analysis virtually eliminates all the discrepancies between the measured and the calculated energies of the $5g$-$4f$ transitions in muonic atoms of heavy nuclei.

There appeared to be discrepancies between the measured[1,2] and the calculated[3-11] energies of the $5g$-$4f$ transitions in muonic atoms of heavy nuclei: Pb, Tl, and Hg. The measured energies are ~400 keV, while the calculated values are systematically higher by ~50 eV, or 2–3 standard deviations. The fractional discrepancies are roughly 1 part in 10^4, or $\sim\alpha^2$. Since $Z\alpha$ is not a small number for these nuclei, it is desirable, at least in principle, to keep the exact dependence on $Z\alpha$. The dominant correction of ~2 keV comes from the vacuum polarization diagram 1 of order α in Fig. 1. The self-energy diagram 2 of order α^2 has been completely neglected so far.

The order of magnitude of diagram 2 can be seen in the following way. Consider the electron screening diagram 3 and the atomic polarizability diagram 4 of the usual perturbation expansion:

$$H = H_\mu - Z\alpha/r_\mu + H_e - Z\alpha/r_e + \delta H, \qquad \delta H = H_\gamma, \tag{1}$$

where $H_\mu - Z\alpha/r_\mu$ and $H_e - Z\alpha/r_e$ represent the unperturbed muon and electron Hamiltonians, and H_γ represents the electromagnetic interaction between the muon and the electrons. Since the wave functions of the atomic electrons behave more like those in a nucleus of atomic number $Z-1$ rather than Z, a better first approximation is given by the alternative expansion

$$H = H_\mu - Z\alpha/r_\mu + H_e - Z\alpha/r_e + V_\mu(r_e) + \delta H, \qquad \delta H = H_\gamma - V_\mu(r_e), \tag{2}$$

where $V_\mu(r_e)$ represents the electrostatic potential created by the muon. (The fully self-consistent Hartree-Fock Hamiltonian is not necessary for the qualitative discussions given here, since the influence of the electrons on the muonic wave functions is very small.) The probability density of the $1s$ atomic electrons near the origin is roughly proportional to Z^3; therefore, the electron screening correction changes by an amount of the order of $-3/Z$ times the original correction. This change provides a simple estimate for

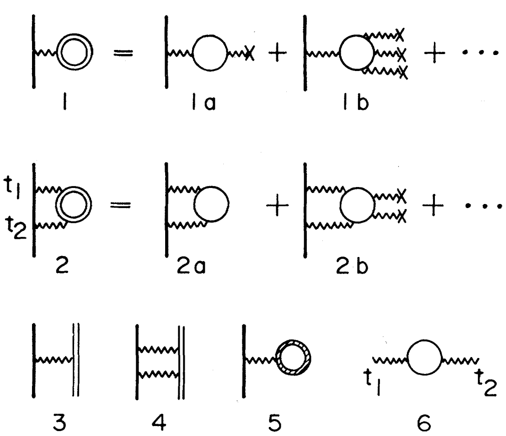

FIG. 1. Radiative corrections. The heavy lines represent the bound muonic states or the exact muon propagators in the Coulomb field of the nucleus. The double lines represent similar states or propagators of the electron. The hatched double line represents the exact electron propagator in the Coulomb field of the nucleus plus the static Coulomb field created by the muon. The single lines represent the free-electron propagators. The crosses represent the nucleus. The photon lines in diagram 2b can be permuted, and such permutations, not shown in the figure, are included in the calculation.