Critical Exponent Inequalities and the Continuity of the Inverse Range of Correlation*

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The inverse range of correlation for nearest-neighbor, ferromagnetic, Ising models is shown to vanish continuously as the temperature approaches the critical temperature. An example is also given where this property fails. The critical index obeys $\nu \geq 2/(d+1)$. Attention is drawn to the relevance of continuity to mass renormalizability in Euclidean boson quantum field theory.

A fundamental assumption underlying the scaling theory[1] of critical phenomena, and *a fortiori* the renormalization-group theory[2] of critical phenomena, is that the range of correlations, ξ, is the single important parameter and that it tends to infinity at the critical temperature. I will prove this assumption for ferromagnetic, nearest-neighbor, Ising models. In consequence, following arguments analogous to those of Glimm and Jaffe[3] for Euclidean $:\varphi_2{}^4:$ quantum field theory, I am able to bound the correlation-length critical exponent $\nu \geq 2/(d+1)$ and the susceptibility exponent $\gamma \geq 1$.

In the case of Euclidean, lattice, boson field theory, ξ^{-1} is directly analogous to the renormalized mass m. The continuity of ξ^{-1} with respect to the temperature is the analog of the continuity of m with respect to the bare mass. This key step is required to establish mass renormalizability. A fuller exposition of this subject including mass renormalizability for $:\varphi_d{}^4:$ will be given elsewhere.[4]

The continuity of the inverse range of correlation is not just a mathematical refinement, but appears to be related to the short-range nature of the interactions. I give the following example where the inclusion of a long-range interaction causes ξ^{-1} to be discontinuous at the critical point. The Hamiltonian is[5]

$$H = -J \sum_{\substack{\text{nearest}\\ \text{neighbors}}} \sigma_i \sigma_j - b\left(\textstyle\sum_i \sigma_i\right)^2/N, \tag{1}$$

where $\sigma_i = \pm 1$, N is the number of spins, and the spins are on a plane square lattice. The second term is a "mean-field"-type interaction. The critical temperature is the solution of the equation

$$2b\chi(\beta_c J) = 1, \tag{2}$$

where χ is the standard, reduced, magnetic susceptibility.[6] The inverse correlation length is that implied by the first term (two-dimensional, Ising model) for $T > T_c$ and drops discontinuously to zero at $T = T_c$.

In order to guide considerations, it is convenient to consider the spin-spin correlation function for the Gaussian model. [Spins are distributed as $\exp(-\frac{1}{2}\sigma^2)$ instead of $\sigma = \pm 1$ for the Ising model.] Joyce[7] gives, for large separations in d dimensions,

$$\langle \sigma_{\vec{a}} \sigma_{\vec{b}} \rangle \approx \frac{\beta J(|\vec{a}-\vec{b}|/\xi)^{(d-1)/2}}{(2\pi)^{1/2d}|\vec{a}-\vec{b}|^{d-2}} K_{1/2(d-1)}(|\vec{a}-\vec{b}|/\xi) \approx \frac{\beta J \xi^{(3-d)/2}\exp(-|\vec{a}-\vec{b}|/\xi)}{2^{1/2(d+1)}\pi^{(d-1)/2}|\vec{a}-\vec{b}|^{(d-1)/2}}, \tag{3}$$

where $K_l(x)$ is a modified Bessel function of the second kind, and

$$\xi = [\beta J/(1-2d\beta J)]^{1/2}. \tag{4}$$

We are now in a position to introduce the definition of ξ^{-1} for a finite system of N^d spins. It is essential to include the asymptotic power law as well as exponential aspects of the correlation function. An arbitrary coefficient which goes to zero with ξ^{-1} is included in the power law term. Use free boundary conditions,

$$\xi^{-1} = \min_{\vec{r},\vec{s}} \left\{ \xi^{-1} \,\middle|\, 0 \leq \langle \sigma_{\vec{r}} \sigma_{\vec{s}} \rangle \leq \frac{e^{-|\vec{r}-\vec{s}|/\xi}}{1+\xi^{-\alpha}|r-s|^{(d-1)/2}} \right\}, \tag{5}$$

where $\alpha>0$ is a temperature-independent constant, and compute

$$\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle=\sum_{\{\sigma_{\vec{u}}=\pm1\}}\sigma_{\vec{r}}\sigma_{\vec{s}}\exp\Big(\beta J{\sum_{\vec{u},\vec{v}}}'\sigma_{\vec{u}}\sigma_{\vec{v}}\Big)\Big/\sum_{\{\sigma_{\vec{u}}=\pm1\}}\exp\Big(\beta J{\sum_{\vec{u},\vec{v}}}'\sigma_{\vec{u},\vec{v}}\Big) \tag{6}$$

with ${\sum'_{\vec{u},\vec{v}}}$ the sum over a d-dimensional space lattice and $\vec{u},\vec{v}$ are nearest-neighbor sites. Since

$$f(x)=e^{-xA}[1+x^{\alpha}B]^{-1} \tag{7}$$

is monotonic decreasing from $f(0)=1$ to $f(\infty)=0$ for real nonnegative x, and $0\le\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle\le1$ by the magnitude of the σ's and Griffith's inequality,[8] Eq. (5) defines a value of ξ^{-1} in the range $0\le\xi^{-1}<\infty$. The value so defined decreases monotonically as the system size N increases. To see this adjoin a hyperlayer of uncoupled spins; then $\langle\sigma_{\vec{r}}\sigma_{\vec{N+1}}\rangle=0$, which will not be the minimum in Eq. (5). If we now couple these spins into the lattice then, by the Griffiths[8] and Kelly and Sherman[9] inequalities, every spin-spin correlation function increases, and so

$$\xi^{-1}(N+1)\le\xi^{-1}(N). \tag{8}$$

Thus, in the limit as $N\to\infty$

$$\xi^{-1}=\lim_{N\to\infty}\xi^{-1}(N) \tag{9}$$

as $\xi^{-1}\ge0$. I remark that the only way that ξ^{-1} can approach zero is if either $\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle=1$ for all $\vec{r},\vec{s}$ [by the Griffiths-Kelly-Sherman (GKS) inequalities] or $|\vec{r}-\vec{s}|\to\infty$, by the structure of the definition. The definition of ξ^{-1} has not been proven to be the same as that of the true inverse range of correlation ξ_T^{-1}. It is true at least that $\xi_T^{-1}\ge\xi^{-1}$, and that they go to zero together, although the rate may possibly not be the same.

Following Glimm and Jaffe[3] my aim is to establish a uniform bound on the derivative of ξ^{-1} with respect to $K=\beta J$, the temperature variable. For N finite, ξ^{-1} is a continuous, piecewise differentiable function of K. By direct calculation from Eq. (5), we have, differentiating the minimum term for finite N,

$$0\le-\frac{\partial\xi^{-1}}{\partial K}\le{\sum_{\vec{u},\vec{v}}}'\frac{\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\sigma_{\vec{u}}\sigma_{\vec{v}}\rangle-\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle\langle\sigma_{\vec{u}}\sigma_{\vec{v}}\rangle}{\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle|\vec{r}-\vec{s}|}, \tag{10}$$

where use has been made of the GKS inequalities to prove positivity, and a factor multiplying the derivative which depended on $|\vec{r}-\vec{s}|$ and ξ^{-1} was bounded from below. By use of one of a general set of inequalities due to Lebowitz,[10]

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle-\langle\sigma_1\sigma_2\rangle\langle\sigma_3\sigma_4\rangle\le\langle\sigma_1\sigma_3\rangle\langle\sigma_2\sigma_4\rangle+\langle\sigma_1\sigma_4\rangle\langle\sigma_2\sigma_3\rangle, \tag{11}$$

we can bound Eq. (10) by

$$-\frac{\partial\xi^{-1}}{\partial K}\le{\sum_{\vec{u},\vec{v}}}'\frac{\langle\sigma_{\vec{r}}\sigma_{\vec{u}}\rangle\langle\sigma_{\vec{s}}\sigma_{\vec{v}}\rangle+\langle\sigma_{\vec{r}}\sigma_{\vec{v}}\rangle\langle\sigma_{\vec{s}}\sigma_{\vec{u}}\rangle}{\langle\sigma_{\vec{r}}\sigma_{\vec{s}}\rangle|\vec{r}-\vec{s}|}. \tag{12}$$

Without significance to the argument, we set $\vec{u}=\vec{v}$ since they are nearest-neighbor sites. Noticing that the inequality in (5) is an equality for the minimum $(\vec{r},\vec{s})$ we may then bound (12) by

$$0\le-\frac{\partial\xi^{-1}}{\partial K}\le q\sum_{\vec{t}}\frac{[1+\xi^{-\alpha}|\vec{r}-\vec{s}|^{(d-1)/2}]\exp[\xi^{-1}(|\vec{r}-\vec{s}|-|\vec{r}-\vec{t}|-|\vec{s}-\vec{t}|)]}{[1+\xi^{-\alpha}|\vec{r}-\vec{t}|^{(d-1)/2}][1+\xi^{-\alpha}|\vec{s}-\vec{t}|^{(d-1)/2}]}, \tag{13}$$

where q is the lattice coordination number. If $|\vec{r}-\vec{s}|$ is not large, then Eq. (13) gives an immediate finite bound. If $|\vec{r}-\vec{s}|$ tends to infinity with N, a little calculation is required. Let us consider the hyperellipsoids on which the argument of the exponential in Eq. (13) is constant. They have their foci at $\vec{r}$ and $\vec{s}$. If we parametrize them by ζ, the length of the semiminor axes, then the semimajor axis is

$$(\zeta^2+\tfrac{1}{4}|\vec{r}-\vec{s}|^2)^{1/2}\approx\tfrac{1}{2}|\vec{r}-\vec{s}|+\frac{\zeta^2}{|\vec{r}-\vec{s}|}+\ldots, \tag{14}$$

where the right-hand side is valid if $\zeta\ll|\vec{r}-\vec{s}|$. In the limit of large but finite $|\vec{r}-\vec{s}|$, we may replace summation in (13) by integration and integrate first along the direction of the major axis of the hyperellipsoid. We may reduce the dominant term in (14) to be proportional to

$$\xi^{\alpha}\int_0^{\infty}\zeta^{d-2}\exp\left(\frac{-2\xi^{-1}\zeta^2}{|\vec{r}-\vec{s}|}\right)\frac{d\zeta}{|\vec{r}-\vec{s}|^{(d-1)/2}}, \qquad (15)$$

where the constant of proportionality is dimension dependent and use has been made of the decrease in the volume of a hypercylindrical slice near the ends of the hyperellipsoid to offset the decrease in the denominator in Eq. (13). The form (15) is correct for $d \geq 2$; for $d=1$ the integration does not occur and the integral is replaced by a constant. I conclude, integrating (15), that

$$0 \leq -\partial\xi^{-1}/\partial K \leq \Gamma_d \xi^{\alpha+(d-1)/2}, \qquad (16)$$

or

$$0 \leq -\partial(\xi^{-1})^{\alpha+(d+1)/2}/\partial K \leq \Gamma_d, \qquad (17)$$

$\alpha > 0$, uniformly in N, and K. Thus, Eq. (17) implies, in the limit as $N\to\infty$, the continuity of a positive power of ξ^{-1} and hence of ξ^{-1} itself. Now I can exhibit for sufficiently high temperature, by series methods, a nonzero ξ^{-1}. By Onsager's solution[11] for the two-dimensional Ising model and by the GKS inequalities[8,9] there are low enough temperatures for which long-range order exists, which forces $\xi^{-1}\to 0$ by our definition. It follows therefore from (17) that there is a critical value of $K<\infty$ for $d\geq 2$ such that

$$\xi^{-1} \leq [\Gamma_d(K_c - K)]^{1/[\alpha+(d+1)/2]}. \qquad (18)$$

Thus I conclude

$$\nu \geq 2/(d+1) \qquad (19)$$

as (18) holds for any $\alpha > 0$. This relation is obeyed by the known exact results $\nu = 1$, $d=2$, and by the numerical estimates $\nu = 0.64$, $d=3$ and $\nu \simeq 0.5$, $d \geq 4$. While (18) does not hold with $\alpha = 0$, (19) does as the standard definition of an exponent is not changed by, for example, logarithmic correction terms. The bound on index γ for the susceptibility χ follows by a short argument due to Jaffe[12] which applies here as well. By definition,

$$\begin{aligned} d\chi/dK &= \sum_{\vec{u},\vec{v}}{}' \sum_{\vec{t}} [\langle\sigma_{\vec{u}}\sigma_{\vec{v}}\sigma_{\vec{s}}\sigma_{\vec{t}}\rangle - \langle\sigma_{\vec{u}}\sigma_{\vec{v}}\rangle\langle\sigma_{\vec{s}}\sigma_{\vec{t}}\rangle] \\ &\leq \sum_{\vec{u},\vec{v}}{}' \sum_{\vec{t}} [\langle\sigma_{\vec{u}}\sigma_{\vec{s}}\rangle\langle\sigma_{\vec{v}}\sigma_{\vec{t}}\rangle + \langle\sigma_{\vec{u}}\sigma_{\vec{t}}\rangle\langle\sigma_{\vec{v}}\sigma_{\vec{s}}\rangle] \end{aligned} \qquad (20)$$

by inequality (11). This, by summing first over $\vec{t}$, becomes, if q is the coordination number of the lattice,

$$0 \leq d\chi/dK \leq q\chi^2, \qquad (21)$$

which implies $\gamma \geq 1$, if $\chi\to\infty$ as $\xi^{-1}\to 0$.

The arguments given can be extended to general discrete spin, and also to continuous spin with weight functions of the nature $\exp(-aS^4 + bS^2)$ by the method of Simon and Griffiths.[13] This extention is important in the field-theory case.[4]

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