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§Now at Clinton P. Anderson Meson Physics Facility, Los Alamos, N. M. 87544.

[1]Definitions of the form-factor formalism and the experimental status of K-meson form-factor measurements can be found in N. Barash-Schmidt *et al.*, Lawrence Berkeley Laboratory Report No. LBL-100, 1974 (unpublished), pp. xi, 51, 57.

[2]H. Hinterberger *et al.*, Rev. Sci. Instrum. 41, 413 (1970).

[3]A water Cherenkov counter designed by D. Hutchinson.

[4]The time distribution of the proton bunch in the Princeton-Pennsylvania Accelerator was not known but is simulated by a Gaussian jitter introduced into the Monte Carlo simulation. Many details of these experiments and the Monte Carlo calculation appear in J. Nagy, doctoral dissertation, University of Pennsylvania, 1974 (unpublished); and in R. Werbeck, doctoral dissertation, University of Pennsylvania, 1973 (unpublished).

[5]V. Bisi *et al.*, Phys. Lett. 36B, 533 (1971).

[6]C.-Y. Chien *et al.*, Phys. Lett. 35B, 261 (1971).

[7]S. Marateck and S. P. Rosen, Princeton-Pennsylvania Accelerator Report No. PPAR-8, 1969 (unpublished).

[8]S. Gjesdal *et al.*, as cited in V. Soergel, J. Phys. (Paris), Colloq. 34, C1-83 (1973), p. 85.

[9]R. A. Zdanis *et al.*, as cited in V. Soergel, J. Phys. (Paris), Colloq. 34, C1-83 (1973), p. 86.

Spectrum of Heavy Mesons in e^+e^- Annihilation*

Barry J. Harrington, Soo Yong Park, and Asim Yildiz
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
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The picture that charmed quarks are interacting with each other through a linear potential is used to analyze the particle spectrum in the high-energy annihilation process $e^+e^- \rightarrow$ hadrons. With the newly found resonances at 3.1 and 3.7 GeV as input, we predict other resonances (in particular, one at approximately 4.0 GeV) and determine the hadronic width of the 3.1 and leptonic widths of the 3.1 and 3.7 resonances. Extending the linear potential to quarks without charm enables us to predict the charmed-particle threshold to be approximately 5 GeV.

Recent experiments concerning $e^+e^- \rightarrow$ hadrons are providing a particularly instructive probe of hadronic structure and dynamics.[1] Especially interesting are the newly discovered narrow resonances at 3.1 and 3.7 GeV.[2] It has been argued[3] that the 3.1-GeV state is a $^3S_1$ mesonic bound state of charmed[4] quarks, named "orthocharmonium."[5] We speculate that the 3.7-GeV state is a radial excitation of orthocharmonium and is only the first of a new spectrum of charmonium states.[6]

Explicitly, we study a quark system with exact gauge symmetry [e.g., color SU(3)] in which asymptotic freedom is realized in the short-distance limit, as in the scaling region of deep inelastic scattering.[8] In accordance with infrared slavery,[9] the gauge coupling grows with the quark separation, bring us into the domain described by a strong-coupling theory. Here, we use Schwinger[10] and Wilson[11] as guides and picture the quarks as confined by a flux of a gauge field, which leads to a linearly rising energy between the quarks.

These ideas are applied to $e^+e^- \rightarrow$ hadrons by considering the fate of the colored quark and antiquark produced from the single-photon remnant of the e^+e^- annihilation. Initially, the quark (q) and antiquark ($\bar{q}$) recede from each other almost freely, in accord with the ideas of asymptotic freedom. However, as the separation increases, a linear potential emerges and dominates. Now if the quarks do not fall into an energy eigenstate of the linear potential, they undergo a bremsstrahlung-type radiation with soft gauge photons (massless vector gluons). These gauge photons create a shower of quark pairs which form the final-state hadrons, as in the quark-parton explanation of scaling. But, if the quark energy coincides with an energy eigenvalue of the linear potential, then the quark is prohibited from producing the bremsstrahlung radiation and a bound state is formed. Since the $q\bar{q}$ bound state is formed through a single-photon channel ($J^{PC} = 1^{--}$), it must be a spin triplet, which we call orthocharmonium.[5] We assume that insofar as the quarks are in an energy eigenstate, their lifetime is sufficiently long so that their states are completely characterized by a linear potential. In addition, a single-particle wave-function description should be adequate, for the bound states are free from the bremsstrahlung type of pair creation. As in the usual quark bound-state calculations,[12] we use the nonrelativistic Schrödinger equation, and, where meaningful, we will determine the relativistic corrections. In the center-of-mass system of $q\bar{q}$, the equation for the radial wave function is

$$\left[\frac{1}{m_{\wp'}}\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{l(l+1)}{r^2}\right)+(Kr-E)\right]\times R(r)=0, \qquad (1)$$

where the potential is taken as $V(r)=Kr$, K being a constant, and $m_{\wp'}$ is the mass of the charmed quark.

The $q\bar{q}$ system produced through a single-photon channel can have only $l=0, 2$ in a spin triplet configuration. For $l=0$, Eq. (1) can be exactly solved and the energy eigenvalues are determined by the zeros of the Airy function,[13] which, when added to the quark rest mass, give the masses of the bound states $M^{(n)}$:

$$M^{(n)}=2m_{\wp'}+(K^2/m_{\wp'})^{1/3}a_n, \qquad (2)$$

where $-a_n$ is the nth zero of the Airy function Ai(x) on the negative real axis. Since we have two parameters m and K, we need two inputs, which we take as $M^{(1)}=3.105$ GeV and $M^{(2)}=3.695$ GeV.[2] Using this, we find $m_{\wp'}=1.16$ GeV while $K=0.211$ GeV^2. It is interesting to note that a similar mass for a charmed quark was considered by De Rújula and Glashow[3] using a different approach.

Now that all parameters are determined, we can calculate the higher resonance states (see Table I). Indeed, the strongest prediction of the linear potential model is the existence of resonances beyond 4 GeV. This is in contrast to the Coulomb potential where, if the first two energy eigenstates are at 3.1 and 3.7 GeV, the continuum will begin at 3.9 GeV. In addition, the rather large values of the mean radius $\langle r\rangle$ would seem to argue against the dominance of the Coulomb potential, particularly for the excited states. Relativistic corrections do not vitiate these results.

TABLE I. Charmonium spectrum. Total energy $M^{(n)}$, with relativistic corrections $\Delta M_{\rm rel}^{(n)}$, and mean radius $\langle r_n\rangle$ as determined from the n zeros of the Airy function at $-a_n$.

n	a_n	$M^{(n)}$ (GeV)	$\Delta M_{\rm rel}^{(n)}$ (GeV)	$\langle r_n\rangle$ (GeV^{-1})
1	2.338	3.105[a]	−0.027	2.5
2	4.088	3.695[a]	−0.082	4.4
3	5.521	4.18	−0.15	5.9
4	6.787	4.61	−0.23	7.2
5	7.944	5.00	−0.31	8.5

[a]Input.

The hadronic decay rate of orthocharmonium can be calculated by considering the dominant annihilation channel of $q\bar{q}\to 3\gamma$, where γ is a vector gluon. The annihilation process takes place essentially in an asymptotically free region, so that we have for the hadronic width of the 1^3S_1 state[5]

$$\Gamma_h(3.1)=[\tfrac{5}{18}][\tfrac{16}{9}(\pi^2-9)\alpha_S{}^3/m_{\wp'}{}^2][m_{\wp'}K/4\pi], \qquad (3)$$

where the $\frac{5}{18}$ is a group multiplicity factor for color SU(3), the second factor is identical in form to the orthopositronium result,[14] and the last factor is the square of the linear potential wave function at the origin. With the gauge coupling in the range[3,5] $0.2<\alpha_S<0.3$, we predict that 50 keV $<\Gamma_h(3.1)<160$ keV. With the same parameters, the Coulomb potential would lead to 8 keV $<\Gamma_h{}^{\rm Coul}(3.1)<92$ keV.

A striking property of the wave function for a linear potential is that it is independent of n at the origin. Thus, the hadronic decay width of orthocharmonium serves as a strict lower bound on the decay widths of the excited states. The leptonic decay width of any of the n^3S_1 states will be given by the contribution from the annihilation into a single photon[14]:

$$\Gamma_l=\frac{16\pi}{9}\left(\frac{\alpha}{m_{\wp'}}\right)^2\left[\frac{m_{\wp'}K}{4\pi}\right], \qquad (4)$$

where $\alpha=1/137$. The predicted leptonic decay width is $\Gamma_l\sim 4.3$ keV. In contrast, the Coulomb result with the same parameters is 0.4 keV $<\Gamma_l<1.3$ keV for $0.2<\alpha_S<0.3$. The linear-potential predictions for the hadronic and leptonic decay widths are in agreement with the preliminary experimental estimates.[2,5]

The predicted leptonic decay width can be used to determine the area under the resonance peak in a graph of the cross section versus energy (c.m.). We use

$$\int\sigma_{e^+e^-\to\,\rm any}\,dE=(6\pi^2/M^{(n)})\Gamma_l \qquad (5)$$

to predict that the area under the 3.1-GeV resonance is 10.3 nb GeV while that under the 3.7-GeV resonance is 7.3 nb GeV. A preliminary estimate[15] of the area under the first resonance is

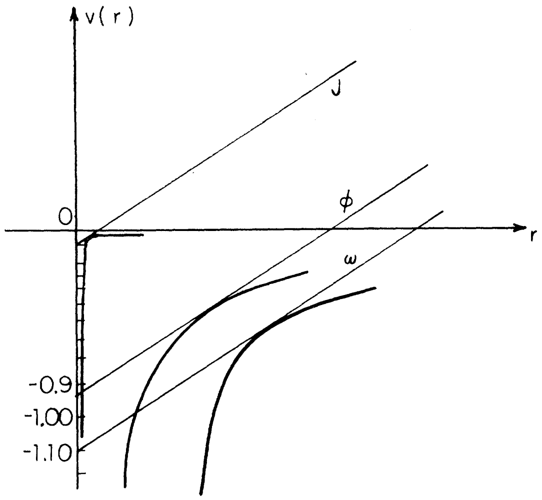

FIG. 1. Potential V as a function of distance r for different total center-of-mass energies.

9 nb GeV.

The procedure to determine the positions and widths of resonances in a linear potential can be used to analyze the spectrum produced by the other quarks, namely the $\mathcal{P}$ (or $\mathcal{N}$) and the λ. Confining ourselves to the mesons with the same quantum number as orthocharmonium restricts our interest to the ω and φ trajectories, and in particular $M_\omega^{(1)} = 0.784$ GeV, $M_\omega^{(2)} = 1.664$ GeV, and $M_\varphi^{(1)} = 1.0196$ GeV. Since these mass values are somewhat smaller than the charmonium masses, we encounter two difficulties. First, the relativistic corrections may become significant, for the binding energies are comparable to those in charmonium but the noncharmed quark masses are smaller. We choose not to forsake the nonrelativistic Schrödinger equation but to consider these relativistic effects further in a future publication. A second difficulty is that we no longer expect the Coulomb potential to be negligible. The physical reason for this can be seen by referring to Fig. 1. As the energy $\sqrt{S}$ of the system of $q\bar{q}$ decreases, the Coulomb coupling, $\alpha(\sqrt{S})$, increases in accord with asymptotic freedom. This implies that the short-distance Coulomb part of the potential becomes less steep. But, if we assume that the linear potential is a smooth outgrowth of the Coulomb potential, then V_0 must become more and more negative as $\sqrt{S}$ decreases. Thus, we modify the linear potential to read $V(r) = V_0 + Kr$, and V_0 will depend on the energy of the ground state of $q\bar{q}$.

For the ω system, using $M_\omega^{(1)}$ and $M_\omega^{(2)}$, we find that $m_{\mathcal{P}} = 0.35$ GeV and $(V_0)_\omega = -1.1$ GeV. The next eigenvalue leads to $M_\omega^{(3)} = 2.38$ GeV. Knowing the light quark mass $m_{\mathcal{P}}$ allows us to predict the threshold for the production of charmed particles to be ~5 GeV. The value of $(V_0)_\omega$ allows us to estimate $(V_0)_\varphi$ (see Fig. 1) for which only $M_\varphi^{(1)}$ is definitely known. Using $(V_0)_\varphi = -1.0$ GeV we find $m_\lambda = 0.47$ GeV while $M_\varphi^{(2)} = 1.8$ GeV. Using $(V_0)_\varphi = -0.9$ GeV we find $m_\lambda = 0.40$ GeV while $M_\varphi^{(2)} = 1.9$ GeV.

Work is now in progress concerning the decay widths of excited states, the hyperfine splitting of orthocharmonium and paracharmonium, as well as other properties of paracharmonium, and the $l \neq 0$ situation.

We are indebted to Sheldon Glashow for suggesting to us the study of the linear potential and for guiding us through this work. We are very grateful to Thomas Appelquist for providing to us both his results and his time for stimulating discussions. We also thank Dr. M. Barnett for a helpful discussion.

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[1]B. Richter, "$e^+e^- \to$ hadron," SLAC-PUB-1478, Plenary Session Report at XVII International Conference on High Energy Physics, London, July 1974.

[2]J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1404 (1974); J. E. Augustin *et al.*, Phys. Rev. Lett. 33, 1406 (1974); C. Bacci *et al.*, Phys. Rev. Lett. 33, 1408 (1974); G. S. Abrams *et al.*, Phys. Rev. Lett. 33, 1453 (1974).

[3]A. De Rújula and S. L. Glashow, Phys. Rev. Lett. 34, 46 (1975).

[4]J. D. Bjorken and S. L. Glashow, Phys. Lett. 11, 255 (1964); S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970); S. Weinberg, Phys. Rev. D 5, 1412 (1972).

[5]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975).

[6]Subsequent to the completion of this work, we became aware of the experimentally observed transition from the 3.7-GeV resonance to the 3.1-GeV resonance with emission of two pions. This fortifies our assumption that the 3.7 GeV is an excited state of the 3.1-GeV resonance.

[7]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973).

[8]H. Georgi and H. D. Politzer, Phys. Rev. D 9, 408 (1974); D. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).

[9]See, for example, S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[10]J. Schwinger, Phys. Rev. 128, 2425 (1962); J. Lowenstein and J. Swieca, Ann. Phys. (New York) 68, 172 (1971); A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974).

[11]K. G. Wilson, Cornell University Reports No. CLNS-262 and No. CLNS-271, 1974 (to be published); J. Ko-

gut and L. Susskind, "Hamiltonian Formulation of Wilson's Lattice Gauge Theories" (to be published); L. Susskind, "Quark Confinement in Gauge Theories of Strong Interactions," in Lectures given at the Bonn Summer School, 1974 (to be published).

[12]J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969).

[13]*Handbook of Mathematical Functions*, edited by M. Abramowitz and J. A. Stegun (National Bureau of Standards, Washington, D. C., 1964).

[14]J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass., 1973), Vol. II.

[15]H. T. Nieh, T. T. Wu, and C. N. Yang, Phys. Rev. Lett. 34, 49 (1975).

COMMENTS

ϵ-Expansion Solution of Wilson's Exact Renormalization-Group Equation*

Geoffrey R. Golner and Eberhard K. Riedel
Department of Physics, Duke University, Durham, North Carolina 27706
(Received 18 November 1974)

The ϵ-expansion formula for the critical exponent η derived by Shukla and Green from Wilson's exact renormalization-group equation is found to yield the standard result $\eta = \frac{1}{54}\epsilon^2 + O(\epsilon^3)$, *independent* of the choice of the incomplete-integration function. The result is obtained analytically in the limit $a \to \infty$ and numerically for arbitrary a, where a denotes the normalization constant of the Gaussian fixed-point functional.

Shukla and Green[1] have studied Wilson's exact renormalization-group equation[2] by using the ϵ-expansion technique about $d = 4$ dimensions. The authors found an expression for the critical exponent η which, they conclude, may violate the universality hypothesis, since it apparently depends on redundant parameters[3] of the calculation. Here we show that their result for η in fact is a universal quantity independent of redundant parameters and equal to the standard result for the continuous-spin Ising model, $\eta = \frac{1}{54}\epsilon^2 + O(\epsilon^3)$.[4,5]

Shukla and Green's ϵ expansion starts from Eq. (11.17) of Ref. 2. If we choose their incomplete-integration cutoff function $\beta(k)$ to have the general form

$$\beta(k) = ck^n, \tag{1}$$

then, at the Gaussian fixed point, u_{20}^* becomes

$$u_{20}^*(k) = \frac{ak^2}{ak^2 + \exp[-(2c/n)k^n]}, \tag{2}$$

where a is the arbitrary kinetic energy normalization. The quantities $\Psi(k)$, $g(k)$, and A defined in Shukla and Green's Letter[1] then become

$$\Psi(k) = (1 + ck^n)\frac{\exp[-(2c/n)k^n]}{\{ak^2 + \exp[-(2c/n)k^n]\}^2}, \tag{3}$$

$$g(k) = \tfrac{1}{2}\{ak^2 + \exp[-(2c/n)k^n]\}^{-1}, \tag{4}$$

$$A = a^2/3\pi^2. \tag{5}$$

Introducing the functions

$$G(k) = 2\int d^4q\, \psi(q) g(|\vec{k} + \vec{q}|) \tag{6}$$

and

$$H(k) = 2\int d^4q\, \psi(q) \int_0^{|\vec{k}+\vec{q}|} \frac{dq'}{q'}[G(q') - G(0)], \tag{7}$$

we obtain for the coefficient of the ϵ^2 term in the expression of the critical exponent η, $\eta = \eta_2\epsilon^2 + O(\epsilon^3)$,

$$\eta_2 = -(A^2/a)H''(0), \tag{8}$$

where $H''(0) = [d^2H(k)/dk^2]_{k=0}$. The question is whether the apparent dependence of this result on the normalization parameter a and the cutoff function $\beta(k)$ is a spurious one.

The discussion is based on the expression for $H''(0)$ below, obtained from Eq. (7) by performing the differentiation with respect to k and the integration over the angular variables:

$$H''(0) = -\pi^2\int_0^\infty dq\, q^2\psi'(q)[G(q) - G(0)]. \tag{9}$$

With transformation to the variable $x = aq^2$, the substitutions

$$\psi'(x) = \hat{\psi}'(x)\, dx/dq, \quad G(q) = (\pi^2/2a^2)\hat{G}(x), \tag{10}$$

with $\hat{G}(0) = 1$ [by definition $G(0) = \frac{1}{6}A^{-1}$], and the partial integration of the $\hat{G}(0)$ term, Eqs. (5), (8),