Evidence Against f-Wave Pairing in Superfluid ^{3}He†

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It is shown that unless strong-coupling corrections to the BCS free energy are very large, the observation by Osheroff and Anderson of a Leggett NMR shift in the A_1 phase of superfluid ^{3}He is incompatible with the hypothesis of f-wave pairing.

Although there is no longer any doubt that a pairing model of superfluid ^{3}He requires the pairs to be spin triplets, their orbital angular momentum has not been as conclusively established. The widespread opinion that the pairs are p waves derives its support primarily from the impressive success of the p-wave model in accounting for some quite intricate and varied magnetic-resonance phenomena. However the angular dependence of the order parameter enters into these explanations only through a relatively small number of its moments. Because of the complexity of models with higher odd L, little effort has been spent to establish that such models could not also yield the necessary structure.[1] Pairing models with $L=3$ are the main rival to the p-wave model, and, for what it is worth, f-wave pairing still emerges as the most favored ordered phase in many microscopic calculations of transition temperatures.[2] To establish the p-wave model more definitively it would help to have, in addition to further explanatory triumphs, at least one piece of direct evidence *against* pairing with $L=3$.[3] In this note I shall describe some recent mathematical results in the theory of f-wave pairing in a strong magnetic field which, when taken in conjunction with the observation by Osheroff and Anderson[4] of a resonance shift in the A_1 phase, provide such evidence, directly giving rise to some grave difficulties for any theory of pairing with $L=3$.

The possibility that the Osheroff-Anderson result might bear on the question of f-wave pairing was known at the time of their experiment. It had been conjectured[5] that the weak-coupling BCS order parameter for an f-wave A_1 phase was proportional to Y_{32}. It had also been pointed out[6] that an order parameter with such a shape has the curious property of yielding *no* Leggett shift in the NMR signal. Osheroff and Anderson's subsequent observation of a shift in the A_1 phase was nevertheless not cited as direct evidence against f-wave pairing for two reasons: (1) The conjecture that Y_{32} minimized the weak-coupling A_1-phase free energy remained unproven; (2) even if the weak-coupling order parameter were proportional to Y_{32} there was no reason to believe that this form would remain stable when strong-coupling corrections were taken into account —indeed, in the complete solution to the analogous, but simpler, $L=2$ problem, it was found that arbitrarily small strong-coupling corrections to the free energy always produced drastic changes in the form of the order parameter.[7]

I have been able to remove the grounds for these reservations by proving (a) that Y_{32} does indeed minimize the weak-coupling f-wave free energy for the A_1 phase and (b) that (in striking contrast to the d-wave case) Y_{32} continues to give the minimum in a substantial neighborhood of the weak-coupling point, in the four-dimensional space of parameters that specify the general fourth-order A_1-phase free energy.

The detailed proof of these assertions will be given elsewhere. This preliminary report is offered because the results shed a new light on the significance of the Osheroff-Anderson experiment, and, in conjunction with the experiment, pose a difficult challenge to those who continue to maintain that "first-principles calculations" favor f-wave pairing. My personal view is that it is most unlikely that strong-coupling corrections can be either large enough or in the right direction[8] to stabilize an f-wave order parameter with the observed A_1 resonance shift. Putting the point another way, should the pairing nevertheless be f-wave pairing and should strong-coupling corrections somehow manage to stabilize (for example) Y_{33} on the melting curve (where the A_1 shift was observed), then, as the pressure dropped and strong-coupling corrections diminished, a first-order transition within the A_1 phase at which the resonance shift abruptly vanished would be quite likely to occur, and it would be surprising if the phase diagram of the A phase were not similarly enriched.

I conclude by displaying enough of the argument to indicate what is meant by "substantial" corrections to the weak-coupling free energy, and to enable anyone with an immediate interest in checking my conclusions to verify them by some rather extensive but straightforward mathematical manipulations.

In the A_1 phase pairing occurs only within a single spin population, and the order parameter is

$$\Delta(\hat{k}) = \langle a_{\vec{p}\uparrow}{}^\dagger a_{-\vec{p}\uparrow}{}^\dagger\rangle|_{\vec{p}=p_F\hat{k}}. \quad (1)$$

Near T_c the form of the f-wave order parameter is determined by minimizing the fourth-order free energy over all normalized linear combinations of spherical harmonics of degree 3. For an f-wave A_1 phase this free energy has the general form[9]

$$f_4 = \sum_{n=1}^{4} \gamma_n f_4^{(n)}, \quad (2)$$

where

$$\begin{aligned} f_4^{(1)} &= [\textstyle\int (d\Omega/4\pi)|\Delta|^2]^2; \\ f_4^{(2)} &= |\textstyle\int (d\Omega/4\pi)\Delta^2|^2; \\ f_4^{(3)} &= \mathrm{tr} M^2, \\ &\quad M_{\mu\nu} = \textstyle\int (d\Omega/4\pi)(\hat{k}_\mu\hat{k}_\nu - \frac{1}{3}\delta_{\mu\nu})|\Delta|^2; \\ f_4^{(4)} &= \mathrm{tr} NN^*, \quad N_{\mu\nu} = \textstyle\int (d\Omega/4\pi)(\hat{k}_\mu\hat{k}_\nu - \frac{1}{3}\delta_{\mu\nu})\Delta^2. \end{aligned} \quad (3)$$

In the weak coupling BCS theory the fourth-order free energy is

$$f_4{}^0 = \gamma^{(0)} \textstyle\int (d\Omega/4\pi)|\Delta|^4. \quad (4)$$

By evaluating both of the forms (2) and (4) for $\Delta = Y_{3m}$, $m = 0, \ldots, 3$, one establishes that in the weak-coupling case the fourth-order parameters are all positive and have the values

$$\begin{aligned} \gamma_1^{(0)} &= \tfrac{210}{143}\gamma^{(0)}, \quad \gamma_2^{(0)} = \tfrac{1}{2}\gamma_1^{(0)}, \\ \gamma_3^{(0)} &= \tfrac{3}{2}\gamma_1^{(0)}, \quad \gamma_4^{(0)} = \tfrac{3}{4}\gamma_1^{(0)}. \end{aligned} \quad (5)$$

Finally, one notes that $f_4^{(1)}$ is normalized to unity, and the $f_4^{(n)}$ cannot be negative. As a result, in the weak-coupling case, *and whenever the* γ_n *have the same signs as in the weak-coupling case,* f_4 cannot be less than γ_1. It is easily verified, however, that with the choice $\Delta = Y_{32}$ this minimum is actually attained [i.e., $f_4^{(m)}(Y_{32}) = 0$, $m = 2, 3, 4$].[10]

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[1]S. Takagi [J. Low Temp. Phys. 18, 309 (1975)] and G. Stare [Ph. D. thesis, Cornell University, 1974 (unpublished)] have shown that the resonance properties of the A_1 and A phases are implied by an order parameter proportional to Y_{LM}, with any odd L, provided that $M^2 > L(L+1)/3$. The resonance properties of the B phase have yet to be accounted for in a model with $L > 1$, but the analysis in this case is exceedingly complex, and no one has yet succeeded in demonstrating that they cannot be so explained.

[2]See, for example, E. Østgaard, Physica (Utrecht) 71, 415 (1974).

[3]G. Barton and M. A. Moore [J. Phys. C: Proc. Phys. Soc., London 8, 970 (1975)] have constructed an intricate and ingenious argument in this direction, but their analysis rests on an f-wave model for the B phase that seems incompatible with the more recent experiments, relies on some unproved mathematical conjectures, and focuses on an experimental result which, if reconfirmed, would prove almost as damning to the p-wave model.

[4]D. D. Osheroff and P. W. Anderson, Phys. Rev. Lett. 33, 686 (1974).

[5]V. Ambegaokar and N. D. Mermin, Phys. Rev. Lett. 30, 81 (1973).

[6]T. Soda and K. Yamazaki, Progr. Theor. Phys. 51, 327 (1974). This follows because to leading order in $|\Delta|^2$, the only second-rank orbital tensor one can construct, $\int d\Omega|\Delta|^2\hat{k}_\mu\hat{k}_\nu$, is proportional to the unit tensor when $\Delta = Y_{32}$, so that the spin-orbit interaction energy is independent of angle when the spin degrees of freedom are rotated with respect to the orbital.

[7]N. D. Mermin, Phys. Rev. A 9, 868 (1974).

[8]D. Rainer and J. Serene (private communication) have calculated that strong-coupling corrections to the f-wave parameters only carry one deeper into the region where Y_{32} gives the stable A_1 phase.

[9]That there are just four independent fourth-order terms follows from an argument such as the group-theoretic analysis of G. Barton and M. A. Moore, J. Phys. C: Proc. Phys. Soc., London 7, 2989 (1974), or from a generalization of the argument given for the d-wave case in Ref. 7. That the four terms given in Eq. (3) are linearly independent follows from evaluating f_4 for Y_{3m}, $m = 0, \ldots, 3$, and noting that the four linear combinations of the γ_n that result are linearly independent. Taking Eq. (3) to give the four fourth-order terms is the key to the result; the rest of the argument requires nothing more than the evaluation of (many) angular averages.

[10]It can be proved that Y_{32} gives the minimum uniquely (to within a rotation of axes). This is not essential for the physical point, however, for any other minimum of f for positive γ_n must also make $f_4^{(3)}$ vanish, which in turn implies (cf. the remark in note 6) that there can be no resonance shift.