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Covariant Dynamical Calculation of the Nucleon-Nucleon *S* Waves*

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We compute the binding energy of the $^3S_1$ and $^1S_0$ NN states (known to be bound by 2.22 and -0.07 MeV) using a covariant, singular-core, three-body model of the $NN\pi$ system with $r_c^{NN}=0.7$ fm, $^3f_\infty=1.8$, and $^1f_\infty=0.3$ as observed at high energy. For the πN P_{11} input we use $r_c^{\pi N}=0.18$ fm fitted (or $\hbar/Mc=0.22$ fm postulated) and find 3.26 (2.59) for $^3S_1$ and 1.41 (0.73) for $^1S_0$.

If we assume that in first approximation nucleon-nucleon scattering can be treated as an $NN\pi$ system below pion production threshold, and that the short-distance (high-momentum) behavior of the NN and πN subsystems can be unambiguously determined from experiment, the binding energy of the $^3S_1$ ("deuteron," $\epsilon_d=2.2$ MeV) and $^1S_0$ ("singlet deuteron," $\epsilon_0=0.07$ MeV) states near the NN elastic-scattering threshold can be predicted. The requisite relativistic three-body formalism has recently been developed by one of us, and successfully applied to the 3π system to show that the ρ generates the ω as the only low-energy $I=0$, 1^-, 3π resonance,[1] as well as to the relativistic πd problem.[2] In the NN system the fact that the N can make a transition to a P_{11} $N\pi$ state with the pion then absorbed by the *other* nucleon gives a type of one-pion-exchange ladder, while the fact that the pion can scatter rather than being absorbed includes two-pion-exchange ladders in the multiple-scattering series summed by the integral equation. To the extent that the *physical* πN amplitude we use reflects "crossed" and "uncrossed" two-pion diagrams and pion-pion scattering (including the "Σ"), we have included these effects without introducing any "renormalization" problems.

The obvious first approximation is to use the nucleon as an s-wave spectator of the πN state which contains the nucleon pole (P_{11}), and the pion as a p-wave spectator of the appropriate NN s-wave ($^1S_0$ to drive the $^3S_1$ calculation and $^3S_1$ to drive the $^1S_0$ calculation). After antisymmetrization in the nucleon variables, our equation takes the form

$$X_i(q_i')=\overline{K}_{i2}(q_i')+\sum_{j=1}^{2}\int_0^{Q_j}dq_j\,q_j^2K_{ij}(q_i',q_j)X_j(q_j). \quad (1)$$

Here X_1 and X_2 represent series of pairwise rescatterings initiated by a πN pair at the nucleon pole; X_1 (X_2) corresponds to a final NN (πN) scattering. The variable q_j is the three-momentum of the spectator particle in the c.m. frame of the pair ($j=1$ corresponds to a spectator pion, $j=2$ to a nucleon). The notation $\overline{K}_{i2}$ represents the residue of K_{i2} at the nucleon pole ($q_2=q_N$); the NN amplitude is given by $t_{NN}=X_2(q_N)$. Because the two-body asymptotic wave functions start right at the singular cores, there is no region in which an extended "potential energy" forces the scatterings "off shell," and all particles are always on mass shell. This explains the one-variable character of the equation, even though the corresponding t matrices are not separable.

The physical justification for using a singular-core model[1,2] to represent the high-energy behavior of our two-body input is the well-known fact that all two-hadron channels can be well approximated by an absorbing disk of constant radius in the particle-production region. Alternatively, we can interpret the boundary condition as approximating a rapid transition from a region where quark degrees of freedom are not much affected by the exterior dynamics to the region of free hadrons, or the stable point a quarter wave outside an internal node in the wave function, which one of us suggested[3] as a way to connect this model to Neudatchin's discussion of internal structure. Either interpretation allows any empirical result to be represented by an energy-dependent logarithmic derivative of the wave function at that radius $\lambda_l(k^2)\equiv[1+l+f_l(k^2)]/r_c$. The two-body amplitudes are then $t_l(k)=N_l(k)/D_l(k)$ with

$$\begin{aligned} N_l(k)&=[r_c\lambda_l(k)-l]j_l(r_ck)+r_ckj_{l+1}(r_ck),\\ D_l(k)&=ik\{[r_c\lambda_l(k)-l]h_l(r_ck)+r_ckh_{l+1}(r_ck)\}.\end{aligned}\tag{2}$$

The energy-dependent function $\lambda_l(k)$ is fitted to scattering data in the physical region $k^2>0$; at large k it is taken to approach the constant value λ_l^∞. Since $\lambda_l(k)$ must be *meromorphic* in k^2 in order for our formalism to produce unitary three-body amplitudes, this fit permits analytic continuation of N_l, D_l to $k^2<0$. Below we use the notation N_l^∞, D_l^∞ to denote N_l, D_l evaluated with $\lambda_l\to\lambda_l^\infty$. The dominant (singular) part of the kernel is

$$\begin{aligned} K_{ij}^s(q_i',q_j)&=\Lambda_{ij}\frac{N_{ij}^s(q_i',q_j)}{D_j(k_j)}\,\frac{N_j(k_j)}{N_j^\infty(k_j)},\\ N_{ij}^s(q_i',q_j)&=-\frac{k_j}{\pi}\int_{-1}^{1}dz\,G_{ij}(z,\hat{K}_{ij}\cdot\hat{q}_j,\hat{Q}_{ij}\cdot\hat{q}_j)\frac{g_i(b_iq_i',b_iQ_{ij})}{q_i'^2-Q_{ij}^2-i\epsilon}\,\frac{N_i^\infty(K_{ij})}{Q_{ij}},\end{aligned}\tag{3}$$

with $\Lambda_{11}=0$, $\Lambda_{21}=\Lambda_{12}=-2/\sqrt{3}$, and $\Lambda_{22}=-\frac{1}{3}$. Although the region where all three particles are close together contributes additional finite terms, which have been given explicitly elsewhere,[1] all the significant dynamics comes from this structure.

In Eq. (3) k_j is the c.m. momentum of the pair. N_j denotes $N_{l(j)}$, where l is the appropriate angular momentum for that pair. The three-vectors $\vec{K}_{ij}$, $\vec{Q}_{ij}$ are the values of $\vec{k}_i$, $\vec{q}_i$ in the i c.m. frame corresponding to $\vec{k}_j$, $\vec{q}_j$ in the j frame (only two such momenta are independent). The function G_{ij} is the geometrical recoupling coefficient which would be unity if all the particles were in relative s waves. As usual in such equations we have a Green's-function denominator corresponding to free propagation,[4] and a factor related to the "off-shell" structure. In our case this is the product $g_iN_i^\infty$, where g_i arises from the sharp cutoff at the pair radii r_c^i (here simplified by using the slightly larger radial parameter b_i). Explicitly

$$g_i(a,b)=ib[bj_\lambda(a)h_{\lambda+1}(b)-aj_{\lambda+1}(a)h_\lambda(b)].\tag{4}$$

Here λ is the angular momentum of the spectator relative to the pair ($\lambda=0$ if $i=2$); note that $g_i(a,a)=1$.

Although the concept of the analytic continuation of two-body amplitudes to obtain dynamical equations is a familiar one, our approach differs from field theory or dispersion theory in that we use the three-body equation to specify the requisite analytic continuation. The Lorentz frame we use for each pair is uniquely specified by requiring that the scattering pair remain in its own c.m. system when the spectator recedes to an infinite distance. This definition reduces to the usual one in the nonrelativistic limit, but introduces important kinematic effects in the covariant evaluation of the quantities $\vec{k}_i$, $\vec{K}_{ij}$, and $\vec{Q}_{ij}$, which enter the above equations. For three particles of mass m_α, m_β, and m_γ treated as free outside the region excluded by the cores, and using a real spectator momentum $q\geqslant0$, the c.m.

energy for the $\beta\gamma$ pair is (with $m=\sum_\alpha m_\alpha$)

$$(m_\beta{}^2+k^2)^{1/2}+(m_\gamma{}^2+k^2)^{1/2}=[s+m^2q^2/(m_\beta+m_\gamma)^2]^{1/2}-[m_\alpha{}^2+m^2q^2/(m_\beta+m_\gamma)^2]^{1/2}, \tag{5}$$

with $s=P^2$, the invariant four-momentum squared of the three-particle system. We see that the upper limit on this energy, and hence on the energy where we need the two-body input for our equation, is achieved at $q^2=0$, while the lower limit, implied by the fact that Eq. (5) can be satisfied only for $k^2 \geqslant -\min(m_\beta{}^2, m_\gamma{}^2)$, fixes an upper limit $q=Q_\alpha$ (infinite if $m_\beta=m_\gamma$). Since the c.m. energy of the $\beta\gamma$ pair is bounded by $\sqrt{s}-m_\alpha$, we see that any three-body treatment of the NN system requires two-body input always a pion mass below the two-body output to be computed. In order to calculate NN scattering near elastic threshold ($\sqrt{s}\sim 2M$), we require only NN input for $-M^2 \leqslant k_{NN}{}^2 \leqslant M\mu(1-\mu/4M)$, and πN input in the narrow range whose upper end is the position of the nucleon pole, $-\mu^2 \leqslant k_{\pi N}{}^2 \leqslant -\mu^2(1-\mu^2/4M^2)$. These amplitudes are obtained from NN and πN scattering data analytically continued to the required region via $\lambda_l(k^2)$, or more conveniently, $f_l=\lambda_l r_c+l+1$. As is common in three-body equations, the left-hand cut structure enters only indirectly through the off-shell behavior, so our requirement that λ_l be a meromorphic function of k^2 makes the extrapolation essentially unique.

For the $^3S_1$ input parameters we use the $^3S_1$-$^3D_1$ coupled-channel fit of Feshbach and Lomon[5] with $r_c{}^{NN}=0.70$ fm, $^3f_\infty=1.8$. We find that by using the same core radius we can fit $^1S_0$ amplitudes up to 1 GeV with the simple parametrization $^1f(k)=0.30-0.27(1+1.16k^2+k^4)^{-1}$ provided we take care to use a coupled-channel formalism above the pion-production threshold and identify the eigenphase with the real part of the elastic-scattering phase. Clearly only the core radius and f_∞ are significant in the kinematic region needed for our calculation, as specified above. By one iteration of the coupled system we can eliminate explicit reference to the X_1 amplitude and isolate the usual one-pion-exchange amplitude as the leading term in t_{NN}. Since the same term would occur if we were calculating higher partial waves, we can identify the coefficient as the constant $G_{np\pi}{}^2$ measured in nucleon-nucleon scattering. This implies, for our simple model, that the residue at the nucleon pole in the P_{11} state is $G_{np\pi}{}^2(1+1.5\mu/M)^{-1}$. It is important to note that the integral in our iterated equation for t_{NN} ends at $Q_2 \simeq m_\pi$ rather than infinity. Therefore we cannot define an equivalent Lippmann-Schwinger equation or NN potential in any meaningful way. Of course we still have a meaningful nucleon-nucleon wave function in coordinate space, and, via the coupled equation, a wave function for the pion coordinate in this system as well. Since the form of the equation is the same for the $^1S_0$ and the $^3S_1$ amplitudes and $r_c{}^{NN}$ is (empirically) the same for both, the splitting we find between these two states comes solely from the difference between $^1f_\infty$ and $^3f_\infty$ in the input. We mirror the tensor force only through the difference between these two parameters empirically observed at *high* (i.e., $T_{\rm lab}{}^{NN} \gtrsim 280$ MeV) energy.

The input for the P_{11} amplitude presents more of a problem, since the nucleon pole is only a pion mass below πN threshold and, in contrast to the NN situation, we are most sensitive to data up to about a pion mass above threshold, where they are poorly known. We know the position of the pole and (as noted above) the residue at that pole in terms of G^2, so the simplest fit has only $r_c{}^{\pi N}$ as a parameter. Using the recent analysis of Carter, Bugg, and Carter[6] we obtain a good fit in the χ^2 sense using all data up to 310 MeV, but for G^2 equal to either 14.6 or 15.3 over half the χ^2 comes from the 310-MeV point where the phase is starting to head toward the Roper resonance. For those who are bothered by our G^2 not being the same in πN and NN scattering, we note that our approach will yield this only when we include antinucleons explicitly; empirically we note that Ball, Shaw, and Wong[7] in fitting P_{11} alone required a smaller value for G^2 than is usually observed either in NN scattering or forward πN-dispersion relations. So we present results for two values of G^2, with and without the highest-energy point. We also use a model with $r_c{}^{\pi N}$ fixed at the value $\hbar/Mc=0.22$ fm and the value which gives about the right binding energy for the deuteron (0.234 fm). More reliable results will have to await a better theoretical understanding of the energy dependence of the P_{11} state, or an accurate value of the scattering length a_{11}, or preferably both.

The results of the calculation are given in Table I. We see that in spite of the uncertainties engendered by the uncertainty in the P_{11} amplitude, the most significant features of the nucleon-nucleon S waves—namely, two bound states close to zero in units of the pion mass and split by approximately 2 MeV—are stably reproduced. Con-

TABLE I. Dependence of the results on $r_c^{\pi N}$ for various assumptions about the P_{11} πN state.

$r_c^{\pi N}$ (fm)	G^2	χ^2	ϵ_d (MeV)	ϵ_0 (MeV)
0.180	14.6	8.8[a]	3.26	1.41
0.186	14.6	4.2[b]	3.14	1.34
0.196	15.3	9.0[a]	2.96	1.10
0.198	15.3	3.0[b]	3.02	1.17
0.220	15.3	···	2.59	0.73
0.234	15.3	···	2.24	0.48

[a]Ten points from Ref. 6.
[b]Nine points from Ref. 6 (see text).

sidering the simplicity of the model employed and its close connection to empirical results found in quite different experiments and hadronic phenomena, we find this close agreement with experiment truly remarkable.

In order to improve on our calculation we must include additional three-particle states. The two states which we think are of next most importance to the $^3S_1$ calculation are P_{13} coupled to an s-wave nucleon spectator and P_{11} coupled to a d-wave spectator; since they are of approximately equal magnitude and of opposite sign, we expect the prediction for ϵ_d to change very little. The one state we would add to the $^1S_0$ calculation is $^3P_0$ with an s-wave pion spectator; this will make a repulsive contribution, and could push the ϵ_0 prediction up to being just virtual. Equation (5) shows that we need only include elastic two-body amplitudes as input for output below the two-pion production threshold. If we go to higher energy or more particles the most important four-body channels will be those in which the system could separate into two "interacting" subsystems. This configuration is dominated by momenta such that at least one πN pair is near the nucleon pole. If it is precisely at the pole, that pair looks like a nucleon, and we are back to the problem already considered. Therefore we expect four-body corrections to be small.

Although we have not "derived" our input parameters from an elementary-particle theory, it is suggestive that the NN radius has to be $\hbar/2m_\pi c$, the usual estimate of where the NN channel gets lost among other hadronic degrees of freedom; the $N\pi$ radius is close to $\hbar/M_N c$, where the problem also becomes ultrarelativistic. In any case, our consistent covariant treatment of the $NN\pi$ system using the singular-core approach to the three-body problem allows us to understand why the nucleon-nucleon S waves have bound states close to NN threshold, and to obtain a reasonable first approximation to them without any adjustment of the input parameters.

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