TABLE II. $E0$ matrix elements (fm^2) for monopole transitions in ^{24}Mg and ^{28}Si.

Nucleus	Present result	(e,e') [a]	Shell model [b]
^{24}Mg	7.3	6.39 ± 0.39	1.47
^{28}Si	7.7	6.8 ± 0.4	0.92

[a]Ref. 11. [b]Ref. 1.

these wave functions cannot describe the deep structure of the angular distribution. The strong disagreement between experimental and calculated values of the $(1d_{5/2})(1d_{3/2})^{-1}$ particle-hole amplitudes is similar to the discrepancy observed with single-nucleon pickup reactions[12] which also indicates the presence of a large $1d_{3/2}$ particle amplitude. Also, the $E0$ matrix elements obtained in the truncated space of Ref. 1 are in poor agreement with the data (Table II).

In conclusion we believe to have demonstrated that the monopole inelastic cross sections allow a remarkably sensitive test of nuclear wave functions. In particular we find strong evidence for significant $1p$ hole and $1d_{3/2}$ particle components in the wave functions of ^{24}Mg and ^{28}Si. We were also able to derive $E0$ matrix elements which show remarkably good agreement with inelastic-electron-scattering results. Possible contributions from two-step processes are currently being investigated.

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Fluctuation Effects on Directional Data*

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The problem addressed is that of determining best values and confidence limits for the amplitude and phase of an unknown harmonic amplitude vector, given a single set of directions measured on a circle. The previously known solution, for the limiting case of small fluctuation, is inapplicable to the great majority of cosmic-ray anisotropy measurements. Solutions are given that are thought to be valid for all cases likely to occur in practice.

The following results were obtained in response to a specific need that arose recently in the study of cosmic-ray directionality above 10^{19} eV, as determined by air-shower observations.[1] It was found almost immediately that they are also useful for reanalyzing a considerable body of accumulated data pertaining to lower air-shower energies. The results will be developed in more detail, and applied to cosmic-ray data from many sources, in forthcoming publications. They are presented here in general form with the thought that applications may be found in other areas of physics and astronomy.

I consider as data a set of N directions $\psi_1, \psi_2, \ldots, \psi_N$ such that $0 \leq \psi \leq 2\pi$. I assume that the individual values ψ_i have negligible error and equal weight. The usual formulas of Fourier analysis define a first-harmonic amplitude r and phase ψ as follows[2]:

$$r=(a^2+b^2)^{1/2},$$

where

$$a=(2/N)\sum_{i=1}^{N}\cos\psi_i, \quad b=(2/N)\sum_{i=1}^{N}\sin\psi_i,$$

and

$$\psi=\begin{cases}\psi' \text{ if } b>0,\ a>0,\\ \psi'+\pi \text{ if } a<0,\\ \psi'+2\pi \text{ if } b<0,\ a>0,\end{cases} \tag{1}$$

where $\psi'=\arctan(b/a)$, $-\pi/2\leq\psi'\leq\pi/2$.

As the first of three alternatives, I suppose the data set to have been drawn from a population of directions distributed uniformly over the interval 0 to 2π. The asymptotic ($N\to\infty$) probability distribution of r (r distribution) for this case was first given by Rayleigh.[3] Since the harmonic amplitude can be represented as the resultant of N coplanar vectors having magnitude 2 and directions ψ_i, there is an exact analogy to the problem of a simple two-dimensional random walk, first posed by Pearson.[4] Pearson's calculation[5] of the r distribution for small values of N has been supplemented by Greenwood and Durand,[6] using an alternative formulation due to Kluyver.[7] Having called attention to the availability of an exact r distribution, in terms of which all that follows could be reformulated, I develop the remaining argument in terms of the much simpler asymptotic distribution, since from the point of view taken here it proves to approximate the exact one adequately, even for values of N as small as 3. The well-known Rayleigh formula for probability of obtaining fractional amplitude greater than r is given by

$$w(>r)=\exp(-k_0), \quad k_0=r^2N/4. \tag{2}$$

As a second alternative, I suppose the data set to have been drawn at random from a population of directions having a given fractional amplitude s, with phase equal to zero.[8] It follows that $\vec{r}=\vec{s}+\vec{x}$, where $\vec{x}$ describes the Rayleigh fluctuation, and that the probability of r in dr, ψ in $d\psi$, is given by

$$p_{r,\psi}\,dr\,d\psi=(N/4\pi)\exp[-N(r^2+s^2-2rs\cos\psi)/4]\,r\,dr\,d\psi, \tag{3}$$

where $-\pi\leq\psi\leq\pi$. Equation (3) is the starting point for derivation of Eq. (7), to follow. It is also useful in itself. If s is given, for example by theory, it predicts the result of a series of observations, each involving measurement of N directions.[9] Integration with respect to r yields the differential probability distribution of phase (ψ distribution)

$$2\pi p_\psi=\exp(-k)\{1+(\pi k)^{1/2}\cos\psi\exp(k\cos^2\psi)[1+L\,\mathrm{erf}(Lk^{1/2}\cos\psi)]\}, \tag{4}$$

where $k=s^2N/4$, $\mathrm{erf}(x)$ is the error function, and

$$L=\begin{cases}+1 \text{ if } -\pi/2\leq\psi\leq\pi/2,\\ -1 \text{ if } -\pi\leq\psi<-\pi/2 \text{ or } \pi/2<\psi\leq\pi.\end{cases} \tag{5}$$

Integration with respect to ψ yields the r distribution

$$p_r=(rN/2)\exp[-N(r^2+s^2)/4]\,I_0(rsN/2), \tag{6}$$

where $I_0(x)$ is the zero-order modified Bessel function. For $s\gg N^{-1/2}$, p_ψ approaches a normal distribution about $\psi=0$ with dispersion equal to $(2k)^{-1/2}$, while p_r approaches a normal distribution about $r=s$ with dispersion equal to $(2/N)^{1/2}$.

As my last alternative I suppose the data set to have been drawn at random from an anisotropic population characterized by $\vec{s}$, whose value is now unknown, the population having been selected at random from an ensemble in which all possible values of $\vec{s}$ (magnitude and phase) are equally represented. Having calculated r and ψ, I inquire as to the probability $p_{s,\theta}$ that the population from which the data were drawn had s in ds and θ in $d\theta$, where θ is measured relative to ψ. According to (3), the various s and θ combinations have relative probability $\exp[-N(s^2-2sr\cos\theta)/4]$. The remaining structure of Eq. (7) is determined by the requirement that $p_{s,\theta}$ be normalized to unity:

$$p_{s,\theta}=[(k_0/\pi)^{1/2}\pi r I_0(k_0/2)]\exp[-N(s^2+\tfrac{1}{2}r^2-2sr\cos\theta)/4]. \tag{7}$$

TABLE I. Column 1: $\langle\xi\rangle$ (expectancy), equal to $[(\pi k_0)^{1/2}\exp(-k_0/2)\times I_0(k_0/2)]^{-1}$. Column 2: ξ_{mp} (most probable ξ), the solution of $x - I_1(2k_0x)/I_0(2k_0x) = 0$. Column 3: σ_ξ (dispersion), equal to $\{\int[\langle\xi\rangle-\xi]^2 p_\xi d\xi\}^{1/2}$. Columns 4, 5: 5% and 95% confidence limits of ξ. Column 6: 68.3% confidence limit of $|\theta|$ (degrees). Column 7: 95% confidence limit of $|\theta|$ (degrees). (Values shown in parentheses are correct mathematically but have no practical utility.)

	Column number						
k_0	1	2	3	4	5	6	7
1/16	2.328	0	1.758	(.183)	5.719	108.1	(168.0)
1/8	1.697	0	1.286	(.134)	4.164	101.3	(165.8)
1/4	1.274	0	.953	(.101)	3.107	91.2	(163.5)
1/2	1.009	0	.738	(.082)	2.416	76.6	(157.9)
1	.875	0	.597	(.078)	1.978	57.4	144.6
2	.857	.831	.489	.107	1.705	37.5	107.3
4	.914	.930	.371	.280	1.523	23.2	53.3
8	.964	.967	.256	.540	1.383	15.1	30.8
16	.983	.984	.178	.689	1.277	10.3	20.6
32	.992	.992	.126	.785	1.199	7.3	14.3
64	.996	.996	.089	.849	1.142	5.1	10.0

Integration with respect to s yields the θ distribution:

$$2\pi p_\theta = [I_0(k_0/2)]^{-1}\exp[k_0(\cos^2\theta - \tfrac{1}{2})][1 + L\,\mathrm{erf}(Lk_0^{1/2}\cos\theta)]. \tag{8}$$

Integration with respect to θ yields the s distribution, which I prefer to write in terms of the ratio $\xi = s/r$:

$$p_\xi = [2(k_0/\pi)^{1/2}/I_0(k_0/2)]\exp[-k_0(\xi^2+\tfrac{1}{2})]I_0(2k_0\xi). \tag{9}$$

It is noteworthy that p_θ and p_ξ, like $w(>r)$ of Eq. (2), depend on the directly measured quantities N and r only in combination as k_0.

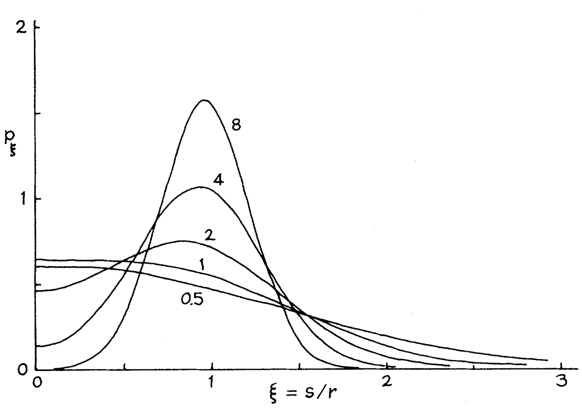

FIG. 1. Differential probability distributions of $\xi = s/r$ labeled with the values of parameter $k_0 = r^2N/4$ to which they belong.

For $k_0 \gg 1$, p_θ approaches a normal distribution about $\theta = 0$ with dispersion equal to $(2k_0)^{-1/2}$, while p_s approaches a normal distribution about $s = r$ with dispersion equal to $(2/N)^{1/2}$. This behavior of p_θ and p_s in the asymptotic limit of small fluctuation could have been predicted on general grounds, as could the corresponding behavior of p_ψ and p_r. Likewise, the asymptotic expressions for dispersion could have been reached by a shorter route. Those expressions have been used for at least two decades in stating results of cosmic-ray anisotropy measurements, often without any apparent regard to the condition $k_0 \gg 1$ on which their validity depends.[10] The behavior of p_ξ is shown in Fig. 1 for a range of k_0 values. Some useful quantities derived from Eqs. (8) and (9) are given in Table I. It can be seen (column

1) that unless $k_0 \gg 1$, r is a biased estimate of s. As k_0 decreases toward unity there is a tendency for r to overestimate s. Fluctuations whose effect is to make r greater than s are more likely than those with the contrary effect. This tendency was noted by Cranshaw *et al.*,[11] who suggested the expression $r[1-1/(2k_0)]^{1/2}$ as an approximation for $\langle s\rangle$. For k_0 values down to 1.5 the approximation is fairly good. Below that it breaks down because of an opposite tendency: For $k_0 \lesssim 0.5$, r tends to underestimate s. Such values of k_0 imply amplitude values so small as to be reckoned unlikely even if s were zero. The amplitude is deviant no matter what one assumes regarding s. Hence such a result has no power to discriminate against s values that are sufficiently small.

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β-Decay Asymmetries in Polarized ^{12}B and ^{12}N and the G-Parity–Nonconserving Weak Interaction

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The decay asymmetries ($\mathcal{A}$) in polarized ^{12}B and ^{12}N have been measured as a function of β-ray energy (E). The coefficients $\alpha_\mp$, in a form of $\mathcal{A} = \mp P(p/E)(1+\alpha_\mp E)$, have been determined to be $\alpha_-(^{12}\mathrm{B}) = +(0.31 \pm 0.06)\%/\mathrm{MeV}$ and $\alpha_+(^{12}\mathrm{N}) = -(0.21 \pm 0.07)\%/\mathrm{MeV}$. The experimental value, $\alpha_- - \alpha_+ = (0.52 \pm 0.09)\%/\mathrm{MeV}$, is larger than the prediction of conserved-vector-current theory, $(\alpha_- - \alpha_+)_{\mathrm{CVC}} \simeq 0.27\%/\mathrm{MeV}$, and in favor of the existence of the second-class induced-tensor current.

Because of the parity nonconservation in the weak interaction, β rays are emitted asymmetrically from polarized nuclei. Conserved-vector-current (CVC) theory predicts, as a result of the weak magnetism, a dependence of the decay asymmetry on the β-ray energy as a higher-or-