$1/f$ Noise—An "Infrared" Phenomenon*

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The current carried by a beam of electrons which have been scattered from an arbitrary potential is shown to exhibit fluctuations with a $1/f$ spectrum up to $f=0$. The exponent of f is smaller than unity by an amount determined by infrared radiative corrections. This amount is essentially the fine-structure constant α and practically coincides with the $1/f$ noise coefficient for one carrier. Similar noise is expected from current carriers in any medium.

Electrical currents transported—e.g., in solids or electrolytes—by a small number of current carriers are known to exhibit low-frequency fluctuations (generally below $f=10^5$ Hz) with a $1/f$ spectrum. This $1/f$ current noise[1] is also known as flicker noise in vacuum tubes, semiconductor devices, or thin films, as excess noise in semiconductors, or as contact noise in poor electric contacts. $1/f$ noise is proportional to the square of the average current, and is inversely proportional to the number of current carriers in the sample. No satisfactory general theory of this universal phenomenon has been presented so far.

Quantum-electrodynamic scattering matrix elements (and cross sections) are finite, i.e., do not exhibit infrared divergences, if coherent states (translated principal vectors) of the electromagnetic field are used.[2] The same finite cross sections are obtained also in the usual Fock representation by adding the virtual- and real-photon contributions in the soft-photon limit,[3] as was shown already by Bloch and Nordsieck. What remains, however, in any case, is a peculiar ϵ^{-1} dependence of the differential cross section on the bremsstrahlung energy loss ϵ.

Consider the scattering of an electron with energy E by a fixed potential. The differential cross section for scattering with an energy loss $\epsilon < E/2$ is given by Eq. 2.57 of Ref. 3:

$$(d\sigma/d\epsilon) = F(\alpha A)\exp[2\alpha(B+\tilde{B})]\{(\alpha A/\epsilon)\tilde{\beta}_0 + G_1(\epsilon)\}. \tag{1}$$

Here σ is the differential cross section $d\sigma'/d\Omega$ for scattering at an angle θ_0 and α is the fine-structure constant, 137^{-1}. The expression in curly brackets is the conventional expression for bremsstrahlung separated into the dk/k contribution and the remainder G_1 which is negligible for $\epsilon \ll E$ and shall be omitted here. $\tilde{\beta}_0$ is the elastic scattering cross section obtained by neglecting all radiative corrections, e.g., the Rutherford cross section when the potential is a Coulomb potential. A is independent of ϵ, and $F(\alpha A)$ is close to unity and of no further importance here. We are interested in small four-momentum transfers ($|q^2| \ll m^2$; $\hbar = c = 1$) for which the infrared exponent is given by Eq. 2.34 of Ref. 3:

$$2\alpha(B+\tilde{B}) \simeq \alpha A \ln\frac{\epsilon}{E} = \frac{2\alpha q^2}{3\pi m^2}\ln\frac{E}{\epsilon} \simeq \frac{8\alpha}{3\pi}\beta^2\left(\sin^2\frac{\theta_0}{2}\right)\ln\frac{\epsilon}{E}, \tag{2}$$

which also defines A. The last form is a nonrelativistic approximation ($\beta = v/c \ll 1$) for small ϵ which allows $\beta' \simeq \beta$. If ϵ_0 is the detection threshold for low-frequency photons, or the lower-frequency limit of the electric-current-noise spectral measurement, we write the integral of Eq. (1) up to an arbitrary $\epsilon_1 \ll E$ in the form

$$\sigma_{\epsilon_1} = F(\alpha A)\tilde{\beta}_0\left[\left(\frac{\epsilon_0}{E}\right)^{\alpha A} + \alpha A\int_{\epsilon_0}^{\epsilon_1}\left(\frac{\epsilon}{E}\right)^{\alpha A}\frac{d\epsilon}{\epsilon}\right]. \tag{3}$$

The two terms in this expression will be observed as elastic scattering and bremsstrahlung up to ϵ_1, respectively. The latter is a coherent superposition of waves with slightly different energies. These yield quantum beats with the elastic wave. The amplitude of the beats is proportional to the square root of the integrand in Eq. (3). Consequently we can see from Eq. (3) directly that the scattered electric current will not be constant and will present fluctuations with a $\epsilon^{\alpha A - 1}$ spectral density, i.e., $1/f$ noise ($0 < \alpha A \ll 1$).

In order to clarify this fundamental observation we write the electron scattered wave within a small solid angle $\Delta\Omega$ far from the scatterer in the simplified plane-wave form

$$\psi = a' \exp[i(\vec{p}\cdot\vec{r} - Et)] + \int_{\epsilon_0}^{\epsilon_1}(d\epsilon/\sqrt{\epsilon})b_\epsilon \exp[i\vec{p}'\cdot\vec{r} - i(E-\epsilon)t]a_\epsilon^\dagger, \tag{4}$$

where, according to Eq. (3),

$$b_\epsilon/a' = (\alpha A)^{1/2}(\epsilon/\epsilon_0)^{\alpha A/2} \equiv \rho_\epsilon, \tag{5}$$

and $a_\epsilon^\dagger$ is a normed superposition of photon creation operators

$$a_\epsilon^\dagger = \epsilon^{-1}\int_{4\pi} f_\epsilon^*(\theta,\varphi)a_{\vec{k}}^\dagger k^2 d\Omega_{\vec{k}}; \quad \int_{4\pi}|f_\epsilon(\theta,\varphi)|^2 d\Omega_{\vec{k}} = 1; \quad k = |\vec{k}| = \epsilon, \tag{6}$$

which corresponds to the emission of a photon of energy ϵ (of indefinite direction angles θ and φ). From

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta(\vec{k} - \vec{k}') \tag{7}$$

we obtain

$$[a_\epsilon, a_{\epsilon'}^\dagger] = (k^2/\epsilon\epsilon')\delta(k-k')\int_{4\pi} f_{\epsilon'}^*(\theta,\varphi)f_\epsilon(\theta,\varphi)d\Omega_{\vec{k}} = \delta(\epsilon - \epsilon'). \tag{8}$$

Note that the scattering matrix element corresponding to Eq. (3) is proportional to $k^{-3/2}$ and that the $\epsilon^{-1/2}$ dependence exhibited by Eq. (4) is obtained after multiplication with $a_{\vec{k}}^\dagger$, after the summation over the final photon states is performed (which yields a factor k^2), and after $a_\epsilon^\dagger$ is introduced [which consumes ak, as we see from Eq. (6)]. $f(\theta,\varphi)$ is proportional to $|\vec{p}/(\vec{k}\cdot\vec{p}) - \vec{p}_i/(\vec{k}\cdot\vec{p}_i)|$ and also includes the phase of the scattering matrix element. $\vec{p}_i$ and $\vec{p}$ are the initial and final four-momenta of the electron and all spin and polarization variables have been suppressed, the electrons being treated classically. I have used the fact that in Eq. (4) $\vec{p}' \simeq \vec{p}$ independent of θ because of the smallness of the energy loss ϵ which shortens $\vec{p}'$ only imperceptibly as compared to $\vec{p}$. In fact, ϵ is so small for $1/f$ noise frequencies (i.e., below 10^5–10^6 Hz) that $(p'-p)\cdot r \ll \pi$ over the measuring region, or over a sample whose $1/f$ noise is measured. Consequently we neglect $\vec{p}' - \vec{p}$, i.e., put $\vec{p}' = \vec{p}$ in Eq. (4), except for the case of feedback amplification, where it may lead to destructive self-interference as the wave ψ runs many times through a closed loop. If integrated over the whole space, all current fluctuations discussed here would disappear because of the orthogonality of electronic wave functions for different, no matter how closely situated, energies. If S is the cross section of the beam and $a = a'\sqrt{S}$ represents a constant, the current corresponding to Eq. (4) will be

$$\vec{j} = \frac{-i}{m}S\psi^\dagger\nabla\psi = \vec{p}\frac{|a|^2}{m}\left(1 + \int_{\epsilon_0}^{\epsilon_1}\rho_\epsilon(e^{i\epsilon t}a_\epsilon^\dagger + e^{-i\epsilon t}a_\epsilon)\frac{d\epsilon}{\sqrt{\epsilon}} + \int_{\epsilon_0}^{\epsilon_1}\frac{de}{\sqrt{\epsilon}}\int_{\epsilon_0}^{\epsilon_1}\frac{d\epsilon'}{\sqrt{\epsilon'}}\rho_\epsilon\rho_{\epsilon'}e^{i(\epsilon'-\epsilon)t}a_\epsilon a_{\epsilon'}^\dagger\right). \tag{9}$$

Denoting by $|0\rangle$ the vacuum state in the Hilbert space of detectable photons ($|\vec{k}| > \epsilon_0$), we obtain for the average current the expression

$$\langle\vec{j}\rangle \equiv \langle 0|\vec{j}|0\rangle = \vec{p}(|a|^2/m)(1 + \int_{\epsilon_0}^{\epsilon_1}\rho_\epsilon^2 d\epsilon/\epsilon) \tag{10}$$

which is in accord with Eq. (3). The fluctuation current is

$$\delta\vec{j} \equiv \vec{j} - \langle\vec{j}\rangle = \vec{p}\frac{|a|^2}{m}\left[\int_{\epsilon_0}^{\epsilon_1}\rho_\epsilon e^{i\epsilon t}a_\epsilon^\dagger\frac{d\epsilon}{\sqrt{\epsilon}} + \text{H.c.} + \int_{\epsilon_0}^{\epsilon_1}\frac{d\epsilon}{\sqrt{\epsilon}}\int_{\epsilon_0}^{\epsilon_1}\frac{d\epsilon'}{\sqrt{\epsilon'}}\rho_\epsilon\rho_{\epsilon'}e^{i(\epsilon'-\epsilon)t}a_\epsilon a_{\epsilon'}^\dagger - \int_{\epsilon_0}^{\epsilon_1}\rho_\epsilon^2\frac{d\epsilon}{\epsilon}\right]. \tag{11}$$

The corresponding autocorrelation function is

$$\tfrac{1}{4}[\langle 0|\delta\vec{j}^\dagger(t+\tau)\delta\vec{j}(t)|0\rangle + \text{c.c.}] = \tfrac{1}{2}\vec{p}^2\frac{|a|^4}{m^2}\int_{\epsilon_0}^{\epsilon_1}\frac{\rho_\epsilon^2}{\epsilon}\cos\epsilon\tau\, d\epsilon. \tag{12}$$

Applying the Wiener-Khintchine theorem and restoring the usual units, we obtain the spectral density of the relative current noise

$$\frac{\langle(\delta j)^2\rangle_f}{\langle j\rangle^2} = \frac{\frac{1}{2}\rho_f^2/f}{[1 + \int_{f_0}^{f_1}\rho_{f'}^2 df'/f']^2} = \tfrac{1}{2}\alpha A\left(\frac{ff_0}{f_1^2}\right)^{\alpha A} f^{-1}. \tag{13}$$

If we integrate from f_0 to f_1 we obtain the total noise

$$\langle(\delta j)^2\rangle = \tfrac{1}{2}(f_0/f_1)^{\alpha A}[1-(f_0/f_1)^{\alpha A}]\langle j\rangle^2 \qquad (14)$$

which has a maximal value of $\langle j\rangle^2/8$, independent of αA, for $(f_0/f_1)^{\alpha A}=\frac{1}{2}$. This suggests that for "halfway resolution" 35% rms $1/f$ noise is associated with the dc current carried by one carrier.

It is important to mention that a $1/f$ noise similar to Eq. (13) is obtained when the incoming particles are described by a density matrix which corresponds to an arbitrary (incoherent) mixture of states with definite energies E_i. This $1/f$ noise is an incoherent sum of $1/f$ noise contributions arising from all components of the density matrix. The classical treatment of the electrons shows that our result applies to any type of charged particles. Consider now the current noise of a small cylindrical semiconductor sample of length l with Ohmic contacts at the ends. Suppose that a nonlocalized electron from a certain point on the Fermi surface of the metal at one side is scattered into a state which carries the current through the semiconductor to the metal on the other side. If we average this state in the semiconductor over the distribution of atomic position coordinates and of atomic and crystal quantum state parameters, we obtain the coherent wave[4] which carries the electron through the semiconductor. This averaged wave includes the effect of multiple scattering in the semiconductor. The coherent wave is defined over the whole semiconductor, and has a form similar to Eq. (4) with the effective mass of the carriers and the medium velocity of light included in Eq. (2). This coherent wave will yield a current similar to Eq. (9) and current noise according to Eqs. (13) and (14). The $1/f$ part represented by the integral in Eq. (4) will not suffer extinction in the averaging process because of the smallness of $(\vec{p}\,'-\vec{p})\cdot\vec{r}\hbar^{-1}$ for $1/f$ noise frequencies.

Denoting by $j_i=\langle j_i\rangle+\delta j_i$ the global current $ev_i^{(x)}$ carried by the carrier i in the axial direction x, we write the relative current fluctuation through the sample in the form

$$\frac{\delta I}{\langle I\rangle}=\frac{(1/l)\Sigma_i\delta j_i}{(1/l)\Sigma_i\langle j_i\rangle}. \qquad (15)$$

If N is the number of carriers in the semiconductor, the relative-current noise spectrum is given in terms of the single-carrier noise spectrum by

$$\frac{\langle(\delta I)^2\rangle_f}{\langle I\rangle^2}=\kappa\frac{\Sigma_i\langle(\delta j_i)^2\rangle_f}{(\Sigma_i\langle j_i\rangle)^2}=\kappa\frac{N\langle(\delta j)^2\rangle_f}{N^2\langle j\rangle}, \qquad (16)$$

where κ is a corrective factor which takes into account the correlations between carriers. For $\kappa=1$ no correlations are present. With the use of Eq. (13) we obtain finally

$$\langle(\delta I)^2\rangle_f=\tfrac{1}{2}\kappa\alpha\bar{A}f^{-1}N^{-1}I^2. \qquad (17)$$

This final result has the form of $1/f$ noise which is observed experimentally.[1,5] $\bar{A}$ is an average over the scattering angle in Eq. (2).

Consider an electric circuit with a bad contact in a certain point P. The bad contact can be represented by a potential which scatters the electrons and P corresponds to the spatial position of the Coulomb potential interaction vertex in the Coulomb scattering example. In the case of $1/f$ noise from a semiconductor sample, the whole sample corresponds in fact to the Coulomb potential and it is only a matter of convenience whether we use part of the interaction to dress the final state. If inelastic scattering is considered, the cross section which replaces Eqs. (1) and (3) turns out[3] to be a convolution of the uncorrected cross section for inelastic scattering with the probability of soft-photon emission. This will again lead to $1/f$ noise in any of the inelastically scattered currents of given energy. When we sum these incoherent $1/f$ noise contributions, we obtain the same result as above.

The vertex corresponding to the emission of a detectable photon is located in the metallic conductors before or after the sample, bad contact, or other obstruction. The phase velocity of low-frequency electromagnetic waves is very small in metals compared with the vacuum (e.g., $10\sqrt{f}$ cm/sec in copper, f in Hz). If we consider that most of the soft-photon emission of a given frequency occurs at carrier velocities close to the corresponding value of the phase velocity of electromagnetic waves in the metal, we are led to set $\beta\approx 1$ in Eq. (2). Taking $\langle\sin^2(\theta_0/2)\rangle=\frac{2}{3}$ and $\kappa=1$, we obtain

$$\tfrac{1}{2}\kappa\alpha\bar{A}\approx 8\alpha/9\pi=2\times10^{-3}. \qquad (18)$$

This coincides with the experimental value obtained by Hooge.[1,5]

The present theory can be considered as a quantized form of the turbulence approach to $1/f$ noise.[6,7] At low frequencies this theory predicts $1/f$ noise for all currents occurring in quantum electrodynamic processes which admit infrared radiative corrections, i.e., all processes involving charged particles. It predicts, e.g., that the small electric currents carried by radioactive $\beta^\pm$

or α emissions will show $1/f$ noise at low frequencies, as is well known for photoelectric currents, or for an electron beam emitted by a hot filament. In a similar way, at sufficiently low frequencies, it predicts $1/f$ noise in neutral beams and macroscopic streams of matter, due to emission of gravitons.

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Nature of $1/f$ Phase Noise*

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The quantum theory of $1/f$ noise developed in the preceding Letter is extended by showing that because of interaction with the electromagnetic field, an alternating current carried by a beam of electrons emerging from some scattering process will present phase fluctuations with a $1/f$ spectrum. These fluctuations are expected in any ac current. They have similar power spectra, also determined by infrared radiative corrections, to dc $1/f$ noise.

$1/f$ noise is known as the most important form of low-frequency electrical noise. In the previous Letter[1] $1/f$ noise has been explained as an intrinsic current-fluctuation process which arises from the interaction of the current carriers with the electromagnetic field. It is known that this interaction associates a probability amplitude for low-frequency photon emission to any scattering process with charged particles. In fact, any dc current involves scattering of charged particles, arising from various interactions of the current carriers. Thus, any current should present some $1/f$ noise. The resultant $1/f$ noise coefficient is inversely proportional to the number of current carriers in the sample.

The emission of low-frequency photons in the frequency range of experimental $1/f$ noise, i.e., below 10^5 Hz, carries an utterly negligible amount of energy, which is uniformly distributed over frequencies down to $f=0$. This uniformity is obvious if we remember that the bremsstrahlung cross section is proportional to df/f and the energy of a photon is hf. Although the energy is negligible, it can be shown that an intrinsic nonstationarity of the current is caused by the low-frequency photon emission.[1] It turns out that each of the df/f photons contributes equally to the current noise spectrum, by "modulating" the current carried by the particle which emitted the photon. Indeed, the average of $\cos^2 2\pi f t$ is $\frac{1}{2}$, independent of f, and a $1/f$ spectrum is the result.

The aim of the present paper is to present another aspect of the same basic phenomenon. Whenever a harmonic high-frequency signal is carried or generated by a finite number of current carriers, a phase-noise measurement will detect phase fluctuations with a $1/f$ spectral density. Consequently, the resulting power spectrum of the electric current will contain quantum noise sidebands with a $1/\Delta f$ spectral density.

In the presence of a harmonic signal of frequency ω_0 the state of a current carrier can be described as a mixture of pure quantum states, among which we also expect states of the form

$$\psi_0 = a\{\exp[i(\vec{p}_1\cdot\vec{r} - E_1 t)] + \exp[i(\vec{p}_2\cdot\vec{r} - E_2 t)]\}, \tag{1}$$

with $E_1 - E_2 = \omega_0$, $\hbar = 1$, $c=1$. Such a split energy state generates an ac current component:

$$\vec{j}_0 = \frac{-i}{m}\psi_0^\dagger \nabla\psi = \frac{|a|^2}{m}\{\vec{p}_1 + \vec{p}_2 + \vec{p}_1\exp[i(\vec{p}_1 - \vec{p}_2)\cdot\vec{r} - i\omega_0 t] + \vec{p}_2\exp[-i(\vec{p}_1 - \vec{p}_2)\cdot\vec{r} + i\omega_0 t]\}, \tag{2}$$