Renormalizability of Paramagnon Theories

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A simple criterion for the renormalizability of paramagnon theories is presented. In contrast to other well-known problems, the paramagnon theory is shown to be renormalizable for the case of a nearly ferromagnetic, three-dimensional fermion system at $T \cong 0$ K.

The purpose of this Letter is twofold: First we want to present a simple criterion for the renormalizability of paramagnon theory, borrowed from the quantized field theory. Secondly, we will use it to give a simple proof that, contrary to most other cases in critical phenomena, the interacting paramagnon theory is renormalizable at $T=0$ K for the three-dimensional (3D) problem of a uniform paramagnetic set of fermions interacting so strongly that the system is close to becoming ferromagnetic at $T=0$ K (the critical temperature is ≤ 0). This implies that the mean-field theory provides a valid starting point to study the enhanced spin fluctuations (paramagnons) in this system.[1] On the contrary, in the case of local paramagnons, for instance, in a metal containing a nearly magnetic impurity (the fermion-fermion spin exchange interaction I appears only at the impurity site), the interactions between the spin fluctuations are so large that the mean field has to be completely modified in describing the behavior of the system near $T=0$ K.[2] The above two problems have been extensively studied.[1-4] However, for some people working in the field of critical phenomena, the renormalizability of the interacting paramagnon theory in the 3D nearly ferromagnetic system appears to be not generally known or even appears puzzling when compared with usual results in critical phenomena. Therefore, we will present here a simple criterion for renormalizability taken from well-known results in quantized field theory and apply this criterion to the case where the basic order parameter is described in terms of a classical field or boson fields. Furthermore, we assume that the fluctuations interact with each other through a short-range potential. Finally, for simplicity, we consider the case where the nonvanishing vertices involve only an even number of interacting fluctuations,[2-4] although the present consideration can be easily extended to a system with elementary vertices having an odd number of fluctuations. The renormalizability of the perturbation series is then established by examining the lowest-order irreducible diagrams associated with the self-energy and the two-fluctuation vertex.[5] The self-energy diagrams were studied in Refs. 1 and 4. Moreover, since the infrared divergence of the self-energy $\Sigma(r)$ is weaker[5] than that of the two-fluctuation vertex $V(r)$ [i.e., the Ward-Takahashi identity states that $V(r) \propto \partial\Sigma(r)/\partial r$, where r is a low-energy cutoff parameter which vanishes at the critical point], we will concentrate on the vertex diagram $V(r)$ shown in Fig. 1. In Fig. 1, the lines are the fluctuation propagators within the mean-field approximation and we sum over all the internal degrees of freedom. Then we can formulate the criterion of renormalizability as follows: If $V(r)$ converges for small r (i.e., if there is no infrared divergence), then the theory is renormalizable. The higher-order corrections to the mean-field quantities are calculated by perturbation in this case. On the other hand, if $V(r)$ diverges, the mean-field theory is meaningless, and the critical region has to be studied from a different starting point. Finally, when $V(r)$ diverges logarithmically, the theory is still renormalizable. In fact, this point

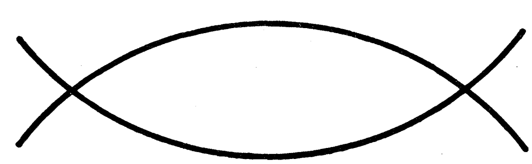

FIG. 1. The diagram calculated in the text as $V(r)$; the two lines are fluctuation propagators.

gives the boundary which separates the region where the mean-field theory is valid and where it becomes meaningless. The celebrated Wilson theory[6] on the $\epsilon = 4 - d$ expansion is based on the fact that in the Wilson model, the dimension $d = 4$ (i.e., $\epsilon = 0$) corresponds to this boundary.

Rigorously speaking, the behavior of higher-order corrections to the fluctuation bubble of Fig. 1 should also be examined, but simple dimensionality arguments makes this study irrelevant, as soon as $V(r)$ converges: Indeed, each time one adds a fluctuation loop, it brings in a factor r^{-2}, but one has also an extra momentum summation which will bring in a factor $r^{d/2}$ in the known Wilson case for instance,[6] so it is still the value of d relative to 4 which matters. In other words, the study of $V(r)$ provides a Ginzburg type of argument to determine the meaningfulness of mean field theory, and the convergence or divergence of $V(r)$ is all we need.

In the following, we will first study the Wilson model. In particular, $V(r)$ for the 4D and 3D Wilson models are explicitly evaluated in (a) and (b) when T approaches T_c from above (i.e., $r = T - T_c \to 0$). In the examples (c) and (d), we examine $V(r)$ for the 3D and 2D, uniform, nearly ferromagnetic fermion systems at $T = 0$ K for vanishingly small $r = E_F - I$, i.e., when the fermion-fermion spin exchange interaction I is close to the characteristic energy E_F of the free fermions. In (e) we study the itinerant ferromagnet above the critical temperature T_c where $r \equiv T - T_c \to 0$. Finally, in (f) we study the local paramagnon case at $T = 0$. In the evaluation of $V(r)$ we introduce the upper cutoff q_0 or ω_c in the integral whenever necessary, since we are not interested in the ultraviolet divergence. Moreover, we drop all numerical coefficients of no interest. When necessary, we will add some comments on the temperature-dependent behavior in the paramagnon cases. However, the temperature in the paramagnon case plays a quite different role compared to that in the Wilson case. This is clearly seen in the cutoff energy r, which measures the distance from the critical point. In the Wilson case $r = T - T_c$, while in the paramagnon case $r = E_F - I$, at $T = 0$ K. In the former case, the critical point is approached from above by decreasing T, while in the latter case by increasing I.

(a) Wilson model (4D).—We have

$$V(r) \propto \int_{|q|<q_0} \frac{d^4q}{(r+q^2)^2} \propto \ln\left(\frac{r+q_0^2}{r}\right) - \frac{q_0^2}{r+q_0^2} \sim \ln r. \qquad (1)$$

Here $V(r)$ diverges logarithmically and the mean-field theory has to be modified slightly. However, the corrections are renormalizable.[6] Another useful way[6] to handle $V(r)$ is to scale q like $r^{1/2}$ so that we have $V(r) \sim r^{4/2-2}$, i.e., $V(r)$ is nondivergent. However, the logarithmic behavior is exhibited in Eq. (1).

(b) Wilson model (3D).—Here

$$V(r) \propto \int_{|q|<q_0} \frac{d^3q}{(r+q^2)^2} \propto \frac{1}{\sqrt{r}} \tan^{-1}\left(\frac{q_0}{\sqrt{r}}\right) - \frac{q_0}{q_0^2 + r} \sim r^{-1/2}. \qquad (2)$$

Now $V(r)$ diverges like $r^{-1/2}$ and the mean-field theory is no longer applicable near the transition temperature where r tends to zero. [Here again, if we scale q like $r^{1/2}$, we have $V(r) \sim r^{1/2}$].

(c) Uniform paramagnons (3D, $T = 0$ K).—Using the well-known paramagnon propagator[7] for small q and small ω/q, we have

$$V(r) \propto \int_{|q|<q_0} d^3q \int_0^\infty \frac{d\omega}{(r+q^2+P_F\omega/q)^2} \propto q_0^2 - r\ln[(q_0^2 + r/r)] \sim \text{const}, \qquad (3)$$

where P_F is the Fermi momentum. $V(r)$ converges at $T = 0$ K, so that the higher-order contributions are completely renormalizable. Actually note that once the ω integral is performed in (3), we are left with

$$V(r) \propto \int_{|q|<q_0} \frac{q\, d^3q}{r+q^2}$$

[i.e., in this 3D problem at $T = 0$ K, $V(r)$ for $I \to E_F$ behaves like $\Sigma(r)$ in the 4D Wilson model for $T \to T_c$, which is known to be convergent]. Therefore, in the present case, the frequency ω plays the role of increasing the effective dimensionality of the problem. Actually ω is not strictly the fourth dimension, as was suggested in Ref. 3, since it appears to the power 1 (corresponding to long-range forces in time), whereas q appears to the power 2 (short-range forces in space). But, as was remarked in Sect. 7 of Ref. 4, q must scale like $r^{1/2}$ but ω/q must scale like r, so ω scales like $r^{3/2}$. One is therefore left with $V(r) \sim r^{(3/2)+(3/2)-2}$; this 3D problem is as renormalizable as the Wilson 4D one, but its "effective" dimensionality is 6 (we thank G. Toulouse for having emphasized the importance of that point), being given by $d_{\text{eff}} = d + 3 = 3 + 3 = 6$ (>4), so that the mean-field result is all the more valid, and the higher-order corrections to the mean-field result can be calculated by perturbation.

We wish here to add some comments about the temperature-dependent behavior of the present problem whose importance will become clearer in connection with case (e). At $T \neq 0$ K (see also Ref. 1),

$$V(r, T) = 2\pi T \sum_{v=-\infty}^{\infty} \int_{|q|<q_0} d^3q (r + q^2 + P_F |\omega_v|/q)^{-2} \propto V(r) + A(T/r)^2, \qquad (4)$$

where $\omega_v = 2\pi T_v$ are the Matsubara boson frequencies and $V(r) \equiv V(r, T=0)$ [Eq. (3)]. Therefore at $T \neq 0$ K, $V(r, T)$ is still nondivergent as long as $T \ll r$. The fluctuation effects enter through r^{-2} in the coefficient of T^2 but can still be handled by perturbation if $T/r \ll 1$; the 3D paramagnon theory is renornalizable at $T=0$ K and for all temperatures such that $T/r \ll 1$. In order to have a nonvanishing region $T/r \ll 1$, r has to be positive. This region shrinks to zero as r tends to zero. We recall that this temperature dependence was extensively studied in Ref. 1. To conclude this case, we note that the mean-field result at $T=0$ K holds in the sense that the critical exponents for the fluctuation propatator are the mean field ones; the renormalized exchange interaction constant $\tilde{I}$ is given in terms of the higher-order corrections to the vertex which are arranged in an ascending power series of r. [In Ma, Béal-Monod, and Fredkin[1], it is shown that the lowest-order perturbation corrections to I due to fluctuations at $T=0$ K give rise to only higher-order terms in $O(r)$ in the presence of a magnetic field; at $T \neq 0$ K, there are corrections of the order of $(T/r)^2$ consistent with Eq. (4).]

(d) Uniform paramagnons (2D, T=0 K).—On substitution of the 2D paramagnon propagator, $V(r)$ becomes

$$V(r) \propto \int_{|q|<q_0} d^2q \int_0^\infty \frac{d\omega}{(r + P_F\omega/q)^2} \propto q_0^3\left(\frac{1}{r} - \frac{1}{r + q_0^2}\right) \sim r^{-1} \qquad (5)$$

and diverges like r^{-1}. (The absence of a q^2 term in the denominator in the integrand is due to the fact that the static susceptibility of the noninteracting fermion system[8] is independent of q up to $q = 2P_F$.) In contrast with the 3D paramagnon case, the mean-field theory does not apply for the 2D one. Note that in the present case the scaling argument used in the previous cases to figure out at once the divergence of $V(r)$ cannot apply here, since q appears in the propagator only in the combination ω/q. However, if we scale q like r to the zero power, while $\omega \sim r$, this yields $V(r) = r^{1-2}$. Therefore, the present case is equivalent to the 2D Wilson model as far as the order of the divergence is concerned; the "effective" dimensionality is identical to the real one.

(e) Itinerant ferromagnet above T_c (finite T_c).—We examine here, for the sake of comparison, the paramagnons in a 3D itinerant ferromagnet above the Curie temperature T_c.[9] In this case we have to evaluate the same integral as defined in Eq. (4), with the only difference that we are concerned in the region $T \gg r$, whereas in (c) we are interested in the region $T \ll r$. In the present case, then, the term with $\omega_v = 0$ (i.e., $v=0$) dominates $V(r)$ and we have

$$V(r) \propto 2\pi T \int_{|q|<q_0} d^3q \frac{1}{(r+q^2)^2} \propto r^{-1/2} \qquad (6)$$

which has the same divergence as the Wilson model (3D) as studied in (b). This is expected from the universality principle for critical behavior. In other words, in the present case the frequency does not help in changing the effective dimensionality in contrast to case (c) and it does not play any role in the static properties. Therefore, in contrast to the nearly ferromagnetic case (c), the mean-field theory does not apply here. We remark here that the mean-field theory in case (c) loses its validity also when the temperature increases so that $T \sim r$. As a conclusion for case (e) we wish to emphasize that since $T_c > 0$, there is no region where $T/r \ll 1$ in contrast to case (c), so that there is no simple connection between the critical behaviors of these two cases.

(f) Local paramagnon, T=0 K.—In this case, the fluctuation propagator depends only on the energy,[2-4] since it corresponds to a local problem. One gets[2-4]

$$V(r) \propto \int_{|\omega|<\omega_c} \frac{d\omega}{(r + |\omega|)^2} = \frac{1}{r} - \frac{1}{r + \omega_c} \sim r^{-1}. \qquad (7)$$

In this case, even at $T=0$ K, $V(r)$ diverges where $r \to 0$, and the problem requires going beyond the mean-field theory. On the basis of a physical argument (that the impurity never becomes magnetic experimentally), it has been demonstrated[2-4] that the fluctuation effects suppress completely the possibility of the formation of a magnetic moment. Rather it was suggested that the critical exponent of the local susceptibility has, most likely, the same value as in the mean-field theory but with a value of the spin exchange interaction I completely modified compared to the mean-field one and, actually, very small compared to it, in contrast with the uniform case (c). Moreover, at finite temperatures,[10] taking for granted the unknown but experimentally measurable $T=0$ K result for the local susceptibility, one can perform a perturbative expansion in powers of T/r, again as long as $T \ll r$, since one has then a small parameter to deal with.

To conclude, we believe that the above comparison between these various cases allows a better understanding of why the 3D uniform paramagnon system at $T=0$ K is renormalizable and thus behaves in an unexpected way. Physically, this may be understood by invoking the same kind of physical arguments we used to make some conjectures for the local case (e): In the uniform paramagnon system at $T=0$ K, the critical point is never really reached since, experimentally, r never becomes strictly equal to zero and the system never becomes ferromagnetic at the lowest available temperatures. Therefore, there is always a region where the perturbation theory works.

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Note added.—After this paper was submitted for publication J. A. Hertz noted at the Twentieth International Conference on Magnetism and Magnetic Materials at San Francisco in December 1974 that whenever $d+z \geq 4$ [where z depends upon the model and is identically 3 in our case (c)], the fixed point of the theory is the Gaussian one, in agreement with our result. However, as we have shown already in case (d), his analysis cannot be carried over to the 2 D case.

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