Excitation of Alfvén Waves by High-Energy Ions in a Tokamak*

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It is shown that shear Alfvén waves can be destabilized by resonance with high-energy "beam" ions near the magnetic axis in a tokamak, if the beam is radially nonuniform.

In a plasma heated by high-energy neutral-beam injection,[1] especially in proposed devices using very powerful injection,[2] the nonequilibrium plasma stability properties must be examined carefully. Previous work has indicated that the distributions that are likely to arise from injection should be stable to velocity-space instabilities in a uniform medium.[3] In particular, for isotropic injection the beam slowing-down distribution is monotonically decreasing in energy, and thus stable to all such modes. Hence, it becomes of interest to look at the modes associated with nonuniform beams and plasmas. In this Letter, we examine the possibility of beam excitation of the shear Alfvén waves. In projected experiments, beam velocities lie just below the Alfvén speed v_A, and hence might be in the proper range for resonant excitation.

As is well known, the dispersion relation for shear Alfvén waves is given by $\omega = k_\parallel v_A \equiv \omega_A$. In tokamak geometry, complications arise in describing the radial eigenmodes since $k_\parallel$ is a function of radius. Thus, for a mode like $\exp(-im\theta + in\zeta)$, where θ is the poloidal angle and ζ the toroidal angle, we have $k_\parallel(r) = (nq - m)/qR$ where $q(r) = rB_\zeta/RB_\theta$; for positive m and n, $k_\parallel(r)$ is an increasing function of r when the current density is centrally peaked. In the magnetohydrodynamic (MHD) approximation, the eigenmodes for the fluid displacement $\xi(r)$ are singular at a radius where $\omega = \omega_A(r)$. To resolve this singularity, it is necessary to include finite gyroradius effects outside the MHD approximation. We will see that these effects also introduce damping of the waves by means of electron dissipation, especially collisions of magnetically trapped electrons. We will also see that the high-energy beam ions can interact resonantly with the waves by means of their ∇B drifts; for nonuniform beams this interaction is destabilizing whenever $\omega < \omega_{*b}$, where ω_{*b} is the diamagnetic frequency of the beam.

Where $\epsilon = v^2/2$ and $\mu = v_\perp^2/2B$, the guiding-center drift equations to an adequate approximation are

$$\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2}, \quad \vec{v}_D = \frac{m(\mu B + v_\parallel^2)}{eB^2}\vec{n}\times\nabla B; \tag{1}$$

$$\vec{v}_p = \frac{m}{eB^2}(1 + \tfrac{3}{4}\rho^2\nabla_\perp^2)\frac{\partial\vec{E}}{\partial t}; \quad \frac{d\epsilon}{dt} = \frac{e}{m}(v_\parallel E_\parallel + \vec{v}_D\cdot\vec{E}); \tag{2}$$

with $\rho = mv_\perp/eB$, the gyroradius. Here $\vec{v}_p$ is the polarization drift including finite gyroradius effects. The drift kinetic equation is then

$$\partial f/\partial t + v_\parallel \vec{n}\cdot\nabla f + (\vec{v}_E + \vec{v}_D)\cdot\nabla f + \nabla\cdot(\vec{v}_p f) + (e/m)(v_\parallel E_\parallel + \vec{v}_D\cdot\vec{E})\partial f/\partial\epsilon = Cf, \tag{3}$$

where we have also introduced a collision operator C. For shear Alfvén waves in a low-β plasma, we can represent the perturbation electric field by

$$\vec{E}_\perp = -\nabla_\perp\varphi, \quad E_\parallel = -\nabla_\parallel\varphi - \partial A_\parallel/\partial t, \tag{4}$$

with $\varphi, A_\parallel \sim \exp(-i\omega t - im\theta + in\zeta)$. In this limit, the magnitudes of B and $\vec{v}_D$ are unaffected by the perturbation. We linearize Eq. (3) about an axisymmetric equilibrium. If we integrate the linearized version of Eq. (3) over all velocities using $d^3v = 2\pi B\, d\mu\, d\epsilon/v_\parallel$, multiply by the charge e, and sum over all species making use of the quasineutrality condition, we obtain a moment equation for $j_\parallel$:

$$\vec{n}_0\cdot\nabla j_{\parallel 1} + \vec{n}_1\cdot\nabla j_{\parallel 0} + \sum e\int d^3v\, \vec{v}_D\cdot\nabla f_1 = -(i\omega n_i m_i/B^2)(1 + \tfrac{3}{4}\rho_i^2)\nabla_\perp^2\varphi, \tag{5}$$

where $\rho_i = (m_i T_i)^{1/2}/eB$. Ultimately, we will combine Eq. (5) with the Maxwell equation $\nabla^2 A_\parallel = -4\pi j_{\parallel 1}$ to yield one relation between φ and $A_\parallel$. This moment-equation procedure makes use of the quasineutrality condition to high order; accordingly relatively small effects, such as the beam contribution to the third term on the left-hand side and the fourth-derivative term, must be retained in Eq. (5). We must proceed to linearize Eq. (3) to obtain the f_1, for use both in Eq. (5) and in the lowest-order quasineutrality condition. We must include a perturbation in the unit vector $\vec{n}$, arising from the magnetic perturbation given by

$$-i\omega\vec{B}_1=\nabla\times(\vec{v}_E\times\vec{B})-\nabla\times(E_{\parallel}\vec{n})=(\vec{B}\cdot\nabla)\vec{v}_E+\vec{n}\times\nabla E_{\parallel}, \tag{6}$$

where we have made use of the fact that the magnitude B of the unperturbed field is approximately uniform. The linearized version of Eq. (3) can then be put in the form

$$(-i\omega+v_{\parallel}\vec{n}\cdot\nabla+\vec{v}_D\cdot\nabla-C)(f_1-\vec{v}_E\cdot\nabla f_0/i\omega)=-\frac{i\omega m f_0}{eB^2}\nabla_\perp{}^2\varphi-\frac{e}{m}(v_{\parallel}E_{\parallel}-\vec{v}_D\cdot\nabla\varphi)\frac{\partial f_0}{\partial\epsilon}+\frac{1}{i\omega}\left[\frac{v_{\parallel}}{B}\vec{n}\times\nabla E_{\parallel}\cdot\nabla f_0-\vec{v}_D\cdot\nabla(\vec{v}_E\cdot\nabla f_0)\right]. \tag{7}$$

Here, we have omitted the fourth-derivative term and have made use of the fact that the collision operator acting on $\vec{v}_E\cdot\nabla f_0$ vanishes, at least in the absence of temperature gradients.

In the case of the electrons, we have $\omega \ll k_{\parallel}v_{\parallel}$. Thus, the terms in $E_{\parallel}$ dominate on the right-hand side in Eq. (7), and to lowest order we have

$$v_{\parallel}\vec{n}\cdot\nabla\left(f_1+\frac{\vec{v}_E\cdot\nabla f_0}{i\omega}\right)=-\frac{e}{m}v_{\parallel}E_{\parallel}\frac{\partial f_0}{\partial\epsilon}+\frac{v_{\parallel}}{i\omega B}\vec{n}\times\nabla E_{\parallel}\cdot\nabla f_0. \tag{8}$$

The second term on the right-hand side in Eq. (8) can be written as $-\omega_{*e}/\omega$ times the first term on the right-hand side, and may thus be neglected since $\omega\gg\omega_{*e}$ for the Alfvén waves of interest. The solution becomes

$$f_1+\frac{\vec{v}_E\cdot\nabla f_0}{i\omega}=g_e=-\frac{e}{m}\frac{\partial f_0}{\partial\epsilon}\int^l E_{\parallel}dl+\bar{g}_e(\mu,\epsilon). \tag{9}$$

Proceeding to next order in $\omega/k_{\parallel}v_{\parallel}$, we must include the terms in $\vec{v}_D$ on the right-hand side in Eq. (7). Again the term in ∇f_0 can be written as $-\omega_{*e}/\omega$ times the term in $\partial f_0/\partial\epsilon$, and may be neglected. In this order we obtain a solubility condition which determines $\bar{g}_e(\mu,\epsilon)$, namely,

$$\oint\frac{dl}{v_{\parallel}}(-i\omega+\vec{v}_D\cdot\nabla-C)g_e=\frac{e}{m}\frac{\partial f_0}{\partial\epsilon}\oint\frac{dl}{v_{\parallel}}\vec{v}_D\cdot\nabla\varphi. \tag{10}$$

For untrapped particles all inhomogeneous terms average to zero ($k_{\parallel}\neq 0$), and we conclude that $\bar{g}_e=0$. For trapped particles, writing $Cg=-\nu_e{}^{\rm eff}g$ and assuming $\omega\gg\vec{v}_D\cdot\nabla$, we obtain

$$\bar{g}_e{}^T=\frac{e}{m}\frac{\partial f_0}{\partial\epsilon}\frac{\omega}{\omega+i\nu_e{}^{\rm eff}}\left\langle\int^l E_{\parallel}dl\right\rangle, \tag{11}$$

where $\langle A\rangle\equiv(\oint A\,dl/v_{\parallel})(\oint dl/v_{\parallel})^{-1}$.

In the case of background ions, we have $\omega\gg k_{\parallel}v_{\parallel}$ and $\omega\gg\nu_i$, and the solution of Eq. (7) is

$$f_1+\frac{\vec{v}_E\cdot\nabla f_0}{i\omega}=g_i=\frac{mf_0}{eB^2}\nabla_\perp{}^2\varphi+\frac{ie}{m\omega}\vec{v}_D\cdot\nabla\varphi\frac{\partial f_0}{\partial\epsilon}. \tag{12}$$

The last term in Eq. (12) is smaller than the second term on the left-hand side by a factor r/R and moreover, being proportional to $\cos\theta$, tends to average out; accordingly, we will omit it.

Finally, we turn to the perturbed distribution function for the high-energy beam ions. For simplicity we will take the beam distribution also to be Maxwellian with $T_b\gg T_{e,i}$. For a Fourier mode like $\exp(in\zeta-im\theta)$, the right-hand side of Eq. (7) may be simplified, and the equation written

$$(-i\omega+v_{\parallel}\vec{n}\cdot\nabla+\vec{v}_D\cdot\nabla)g_b=(ef_0/T_b)(1-\omega_{*b}/\omega)(v_{\parallel}E_{\parallel}-\vec{v}_D\cdot\nabla\varphi), \tag{13}$$

where $\omega_{*b}=-(mT_b/eBr)d\ln n_b/dr$. In tokamak geometry, the particle drift has (r,θ) components given by $\vec{v}_D=[m_b(2\epsilon-\mu B)/eBR](\sin\theta,\cos\theta)$. Also, $E_{\parallel}$ is small and resonance is only possible for $\omega\sim k_\perp v_D\sim k_{\parallel}v_A\gg k_{\parallel}v_b$, so we will neglect the $E_{\parallel}$ term on the right-hand side of Eq. (13).

When we treat the eigenmodes in a sheared field, we will see that the beam resonance is only important in the region $\omega \geq k_{\parallel} v_A > k_{\parallel} v_b$. The angular dependence of $\vec{v}_D$ gives rise to terms like $\exp[in\zeta - i(m \pm 1)\theta]$. Beam resonances can then occur where $\omega = (nq - m + 1)v_{\parallel}/qR$ for positive m and n. The usual case of interest will be where $k_{\parallel}(r)$ is an increasing function of r, since then there will be relatively undamped Alfvén waves occurring between $r=0$ and $r=r_0$, where $\omega = k_{\parallel}(r_0)v_A$. Where v_b now denotes the maximum beam velocity (i.e., the injection speed), the condition for resonance to occur will be $v_b/v_A \geq [nq(r_0) - m][nq(r_0) - m + 1]^{-1}$. Noting that $q(r_0) > q(0)$ and $nq(0) > m$ (or else there would be strong electron damping at the radius where $k_{\parallel} = 0$), this condition on v_b/v_A is not too restrictive, especially for $n=1$, and we will assume in what follows that it is satisfied. In this approximation, the resonant beam term from Eq. (13) is

$$g_b = \frac{\pi i f_0 m_b(2\epsilon - \mu B)}{2BRT_b}\left(1 - \frac{\omega_{*b}}{\omega}\right)\delta\left(\omega - \frac{(nq - m + 1)v_{\parallel}}{qR}\right)\cdot e^{i\theta}\left(\frac{m\varphi}{r} + \frac{\partial\varphi}{\partial r}\right). \tag{14}$$

Next we apply the calculated distribution functions to the macroscopic equations. We first substitute the electron and background ion distributions given in Eqs. (9), (11), and (12) into the lowest-order quasineutrality condition, the beam contribution being negligible to this order. In Eq. (9), we write $\int^l E_{\parallel} dl = E_{\parallel}/ik_{\parallel}$. In the case of interest, we will typically have $(nq - m)\theta_{max} \lesssim 1$, where θ_{max} is the turning point of a typical trapped particle. Accordingly, as a rough average, we may write $\langle \int^l E_{\parallel} dl \rangle = E_{\parallel}/ik_{\parallel}$ in Eq. (11) provided we introduce a factor $(r/R)^{1/2}$ to take into account the number of trapped particles. Quasineutrality then gives

$$E_{\parallel} = -ik_{\parallel}[1 + (r/R)^{1/2}\omega/(\omega + i\nu_e^{\text{eff}})]\rho_{ie}^2\nabla_{\perp}^2\varphi, \tag{15}$$

where $\rho_{ie} = (m_i T_e)^{1/2}/eB$, and $E_{\parallel} = -ik_{\parallel}\varphi + i\omega A_{\parallel}$.

Finally, we combine Eq. (5) and $\nabla^2 A_{\parallel} = -4\pi j_{\parallel 1}$. The beam contributes to the third term on the left-hand side in Eq. (5); we obtain

$$e\int d^3v\, \vec{v}_D\cdot\nabla g_b = -\frac{\pi}{T_b}\left(\frac{m}{2BR}\right)^2\left(1 - \frac{\omega_{*b}}{\omega}\right)\nabla_{\perp}^2\varphi\int f_0(2\epsilon - \mu B)^2\delta\left(\omega - \frac{(nq - m + 1)v_{\parallel}}{qR}\right)d^3v$$

$$\simeq -\frac{n_b T_b}{\omega B^2 R^2}\left(1 - \frac{\omega_{*b}}{\omega}\right)\nabla_{\perp}^2\varphi. \tag{16}$$

Using Eq. (6), and noting that $\partial j_{\parallel 0}/\partial r = -(B/4\pi m r^2)(\partial/\partial r)(r^3\partial k_{\parallel}/\partial r)$, the first two terms in Eq. (5) may be combined and written as $(-i/4\pi r^2)\{(\partial/\partial r)[k_{\parallel}^2 r^3(\partial/\partial r)(A_{\parallel}/rk_{\parallel})] - (m^2 - 1)k_{\parallel}A_{\parallel}\}$. We are now in a position to write down Eq. (5) as a relation between $A_{\parallel}$ and φ, and to substitute for $A_{\parallel}$ in terms of φ using Eq. (15). Changing from φ to a fluid-displacement variable $\xi = -m\varphi/rB$, we obtain

$$\omega^2\rho_i^2(\tfrac{7}{4} - i\delta)\frac{\partial^4\xi}{\partial r^4} + \frac{1}{r^3}\frac{\partial}{\partial r}r^3[\omega^2(1 + i\eta) - \omega_A^2]\frac{\partial\xi}{\partial r} - \frac{m^2 - 1}{r^2}[\omega^2(1 + i\eta) - \omega_A^2]\xi = 0, \tag{17}$$

where $\omega_A(r) = k_{\parallel}(r)v_A$, $\delta = (r/R)^{1/2}\omega\nu_e^{\text{eff}}(\omega^2 + \nu_e^{\text{eff}\,2})^{-1}$, and $\eta = (n_b T_b/n_i m_i \omega^2 R^2)(1 - \omega_{*b}/\omega)$. In obtaining Eq. (17), we have kept only the term $\partial^4\xi/\partial r^4$ out of $\nabla^4\xi$, and have put $\omega^2 = \omega_A^2$ and $T_e = T_i$ in the coefficient of this term.[4]

A complete solution of Eq. (17) is evidently difficult. We observe that away from the singular layer around $\omega = \omega_A(r_0)$, the solution will consist of slowly varying solutions ξ_s and fast-varying (evanescent or oscillatory) solutions ξ_f. We are interested in the case illustrated in Fig. 1, where the fast-varying solution is evanescent for $r > r_0$ and oscillatory within $0 < r < r_0$. Considering the solution only near the singular layer, we may neglect the last term in Eq. (17), and integrate once to obtain an equation for $y = \xi'$, namely, $\rho_i^2(\tfrac{7}{4} - i\delta)y'' + (1 + i\eta - \omega_A^2/\omega^2)y = \text{const}$. The solu-

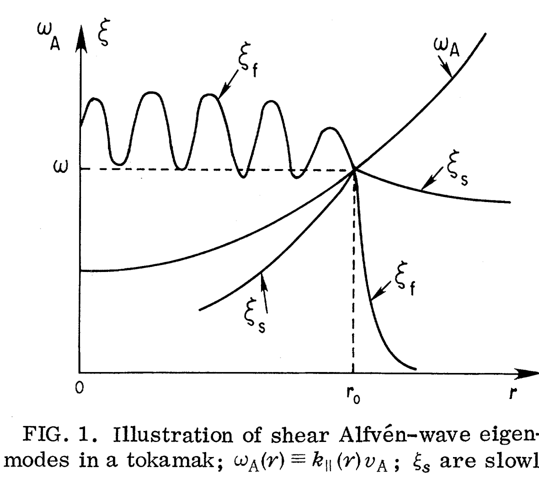

FIG. 1. Illustration of shear Alfvén-wave eigenmodes in a tokamak; $\omega_A(r) \equiv k_{\parallel}(r)v_A$; ξ_s are slowly varying MHD-like solutions; ξ_f are fast-varying evanescent or oscillatory solutions.

tions that decay for $r > r_0$ involve fast-varying oscillatory solutions for $0 < r < r_0$. The condition at $r = 0$ in the present slablike approximation will be $\xi = \xi'' = 0$ (physically, $\xi_r = B_r = 0$). The condition $\xi'' = 0$ will apply to the fast-varying oscillatory part of the solution; considering only this part, we have a WKB condition

$$(\tfrac{7}{4} - i\delta)^{-1/2}\int_0^{r_0}\left(1 + i\eta - \frac{\omega_A^2}{\omega^2}\right)^{1/2} dr = n\pi\rho_i. \quad (18)$$

Marginal stability will occur when the imaginary part of Eq. (18) is satisfied for ω real. We write $\omega_A^2 = \omega_A^2(0) + r^2\partial(\omega_A^2)/\partial r^2$. The marginal stability condition is then $(2\delta r_0^2/7)\partial\ln(\omega_A^2)/\partial r^2 + \eta = 0$. Evidently, for $\omega < \omega_{*b}$ there must always be unstable modes for small enough r_0. Writing $\partial\ln(\omega_A^2)/\partial r^2 = [2m/(nq-m)]\partial\ln q/\partial r^2$, and $n_b(r) = (1 - r^2/2r_b^2)n_b(0)$, the condition $\omega < \omega_{*b}$ becomes

$$\frac{v_b}{v_A}\frac{\rho_b R}{r_b^2}\frac{mq(r_0)}{nq(r_0) - m} > 1, \quad (19)$$

where $\rho_b = (m_b T_b)^{1/2}/eB$. Parenthetically, we note that if $\partial f_b/\partial v_\parallel^2 > 0$ then the condition $\omega < \omega_{*b}$ is no longer required for instability.

In the limit $\omega \ll \omega_{*b}$ the condition for instability becomes [in the case $\omega \gg \nu_e^{\text{eff}}$ with $\nu_e^{\text{eff}} = \nu_e R/r$ and $\nu_e = 2\pi n e^4 \ln\Lambda/m_e^{1/2}(2T_e)^{3/2}$]

$$\frac{r_0^2}{r_b^2} < \frac{\beta_b v_b \rho_b}{\nu_e r_b^4}\left(\frac{r}{R}\right)^{1/2}\left(\frac{\partial \ln q}{\partial r^2}\right)^{-1}\frac{q(r_0)^2}{nq(r_0) - m}, \quad (20)$$

where $\beta_b = 8\pi n_b T_b/B^2$. In applying Eqs. (19) and (20), we note that $nq(r_0) - m$ may be small, although a singular surface where $nq(r) = m$ must not occur. Thus, the worst situation would be where $nq(0) = m$, in which case $nq(r_0) - m = mr_0^2 \times \partial\ln q/\partial r^2$; this case is assumed in the discussion below.

As typical parameters of a large tokamak with intense high-energy injection,[2] we take $B = 50$ kG, $n = 10^{14}$ cm^{-3}, $T = 5$ keV, $\nu_e = 10^4$ sec^{-1}, $v_b = 3\times 10^8$ cm/sec (for 100-keV injection), $v_A = 10^9$ cm/sec, $\rho_b = 0.5$ cm, $R = 300$ cm, $r_b = 50$ cm, and $(\partial\ln q/\partial r^2)^{-1} = 10^4$ cm^2. The instability condition (19) then requires $r_0 < 15$ cm, and for $\beta_b = 0.01$ the condition (20) requires $r_0 < 30$ cm. Thus, the beam may be unstable, and anomalously flattened in radial profile, over the innermost 15 cm. Quasilinear estimates indicate a small slowing down of the beam during this flattening process in the case $\partial f/\partial v_\parallel^2 < 0$. For transient distributions where $\partial f/m\partial v_\parallel^2 \sim +f/E_b$, similar critical values of r_0 are found leading to beam flattening in velocity within this radius. In the case of α particles in a reactor, we might have $B = 50$ kG, $n = 10^{14}$ cm^{-3}, $T = 15$ keV, $\nu_e = 2\times 10^3$ sec^{-1}, $v_b = 10^9$ cm/sec (for 3.5 MeV), $v_A = 10^9$ cm/sec, $\rho_b = 3$ cm, $R = 600$ cm, $r_b = 100$ cm, and $(\partial\ln q/\partial r^2)^{-1} = 4\times 10^4$ cm^2. In this case the instability condition (19) requires $r_0 < 80$ cm, and for $\beta_\alpha = 0.01$ the condition (20) is then also satisfied; this represents a fairly severe condition. The approximations used might lead to errors of a factor 2 in r_0 so that more careful study is justified in some cases.

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[4]In the more general case where the correction factors $1 - \omega_{*i,e}/\omega$ are retained, Eq. (17) is modified as follows: The dispersion function in the square brackets becomes $[\omega(\omega - \omega_{*i}) + i\omega^2\eta - \omega_A^2]$, and the coefficient of $\partial^4\xi/\partial r^4$ becomes $[\frac{3}{4}\omega(\omega - \omega_{*i}) + k_\parallel^2 v_A^2\tau(1 - i\delta)(\omega - \omega_{*i}) \times (\omega - \omega_{*e})^{-1}]\rho_i^2$ with $\tau = T_e/T_i$.