cause the peak for 15 MHz (Ref. 4) is lower than the one for 20 MHz (Ref. 12) by a factor of $\sim\frac{1}{2}$. Similar calculations for the axial state (A phase) are in progress.

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Electron Heating and Landau Damping in Intense Localized Electric Fields*

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A systematic kinetic-theory treatment of the interaction of electrons and ions with intense high-frequency localized electrostatic fields is formulated. A generalization of the familiar nonlinear Schrödinger equation includes nonlinear Landau-damping effects which prevent soliton collapse. An analytic calculation predicts a heated-electron distribution behaving asymptotically as $\exp(-v/v_0)$ ($v_0 = e\mathcal{E}_0/4m\omega_0$) and modulated in the region near the localized field to form streamers in phase space.

Intense, localized high-frequency electrostatic fields appear to be a general feature of strong electrostatic plasma turbulence.[1] Numerical studies of resonance absorption of electromagnetic radiation on an inhomogeneous plasma have shown the existence of intense electrostatic field spikes at the critical density $[\omega_0 = \omega_p(x)]$.[2] Numerical studies of the driven nonlinear Schrödinger equation (NLSE) by Morales, Lee, and White have also shown the evolution of intense electric field solitons which, if pump depletion is neglected, collapse to zero spatial extent.[3] Recently Valeo and Kruer[4] presented results of numerical simulation which demonstrated the obvious importance of wave-particle interactions in stabilizing solitons. Their calculations and a numerical calculation of Morales and Lee[5] show that significantly heated electrons are produced to form high-energy tails in the distribution function. In the present paper we obtain an *analytic* description of the complete electron-velocity distribution function in spatial regions both inside and outside the electric field spike. This description is essential for the correct treatment of Landau-damping corrections to the NLSE which stabilize against soliton collapse. Our work and previous work[1,3-5] is restricted to one dimension.

For this problem we have found it convenient to transform the usual Vlasov equation for $f(v,x',t)$ to a localized oscillating coordinate frame where $v = u + u_H(x,t)$ and $x' = x + x_H(x,t)$ with $u_H(x,t) = -(e/m)\times\int_{-\infty}^{t} dt E_H(x,t)$ and $\partial x_H/\partial t = u_H(x,t)$. Here $E_H(x,t)$ is the localized high-frequency electrostatic field,

$$E_H(x,t) = \tfrac{1}{2}[\mathcal{E}_0(xt)\exp(-i\omega_0 t) + \mathcal{E}_0^*(xt)\exp(i\omega_0 t)]; \quad (1)$$

i.e., $\mathcal{E}_0(x,t)$ is the envelope of the soliton. If $f(v,x,t)$ satisfies the usual Vlasov equation with total electric field $E_{\text{tot}}(x,t) = E_H(x,t) = E_H(x,t) + E_L(x,t)$, where E_L is a low-frequency field, then a new function $F(u - u_H(x,t), x - x_H(x,t); t) = f(u,x,t)$ is readily seen to satisfy the equation

$$\mathcal{L}F(v,x,t) - v(\partial u_H/\partial x)\partial F/\partial v = 0, \quad \mathcal{L} \equiv (\partial/\partial t) + (v\,\partial/\partial x)[(e/m)E_L + \tfrac{1}{2}\,\partial u_H^2/\partial x](\partial/\partial v).$$

In Eq. (1) we have assumed $\partial x_H/\partial x \ll 1$ and $E_H(x + x_H, t) \simeq E_H(x,t)$; i.e., we assume throughout this paper that the excursion distance $x_H(x, t)$ is small on the scale of spatial variations of interest. It is clear from the form of nonlinear terms in this equation that $F(v, x, t)$ must have a high-frequency part (at ω_0) and a low-frequency part: $F(v, x, t) = F_H(v, x, t) + F_L(v, x, t)$. If we average Eq. (1) over the fast-time scale we obtain

$$\langle \mathcal{L} \rangle F_L = \langle v(\partial u_H/\partial x)\partial F_H/\partial v \rangle. \qquad (2)$$

We subtract (2) from (1) to get the fast-time-scale equation

$$\mathcal{L} F_H = v(\partial u_H/\partial x)\partial F_L/\partial v. \qquad (3)$$

The angular-bracket notation denotes the time average over a period $T = 2\pi/\omega_0$ of the high-frequency field. Thus $\langle u_H^2(x, t)\rangle = e^2|\mathcal{E}_0(x, t)|^2(2m^2 \times \omega_0^2)^{-1}$. The characteristics of the homogeneous equation for F_L describe particles moving in an effective low-frequency field which is the sum of $(e/m)E_L = -(e/m)\nabla\varphi$ and the ponderomotive force $\frac{1}{2}\partial\langle u_H^2\rangle/\partial x$.

The low-frequency electrostatic potential satisfies Poisson's equation driven by the low-frequency density. The ion distribution function will be assumed to obey the ordinary Vlasov equation with only the self-consistent low-frequency field E_L. To obtain the equation for the high-frequency field we note that the high-frequency current is $J_H(x, t) = e\int du[uF_H(u, x, t) + u_H(x, t)F_L(u, x, t)]$ and we obtain for the slow-time variation of the envelope function $\mathcal{E}_0(x, t)$

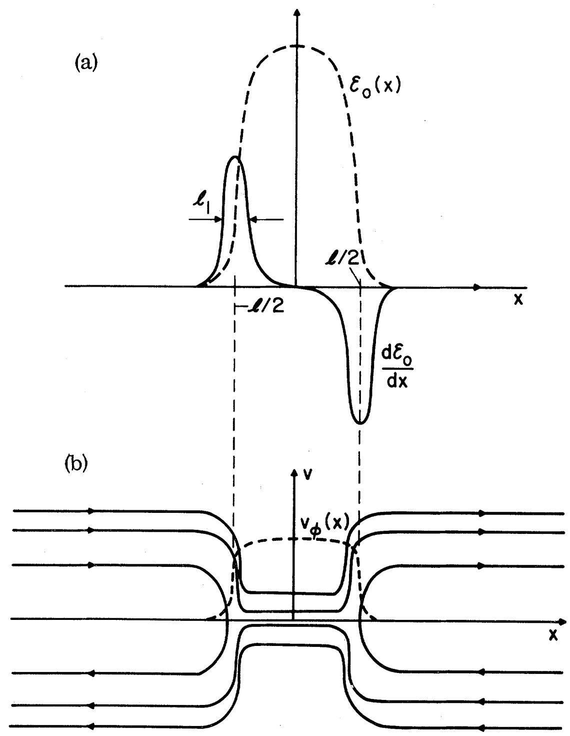

FIG. 1. (a) Schematic shape of soliton. (b) Phase-space trajectories.

$$2i\omega_0\,\partial\mathcal{E}_0/\partial t + \tfrac{3}{2}v_e^2\,\partial^2\mathcal{E}_0/\partial x^2 + (\omega_0^2 - 4\pi e^2/mn_L)\mathcal{E}_0 = \mathrm{Im}[-4\pi e(\partial/\partial t)\int dv\, vF_H], \qquad (4)$$

where $n_L(x, t) = \int du\, F_L(u, x, t)$. The source term on the right-hand side will include Landau-damping effects.

Since we wish to focus on the time dependence of F_H proportional to $\exp(-i\omega_0 t)$ we can replace $u_H^2(x, t)$ in (3) by $\langle u_H^2(x, t)\rangle$, neglecting components which arise at $\pm 2\omega_0$. Then we can formally solve (3) by using the Green's function of the homogeneous equation. We substitute this in (2), write $u_H(x, t) + \frac{1}{2}u_0(x, t)\exp(-i\omega_0 t) + \text{c.c.}$, where $u_0(x, t) = e\mathcal{E}_0/m\omega_0$ is slowly varying in t as is $F_L(v, x, t)$, and carry out the average over the fast-time scale as indicated in (2) to obtain for $v > 0$

$$\partial F_L/\partial t + v\,\partial F_L/\partial x - m^{-1}(\partial/\partial x)(-e\varphi + mu_0^2/4)\partial F_L/\partial v = vS(x, v), \qquad (5)$$

$$vS(x, v) \equiv \tfrac{1}{2}v(\partial u_0/\partial x)(\partial/\partial v)\int_{T_v(x,x')>0} dx' \cos[\omega_0 T_v(x, x')](\partial/\partial_v)F_L(x', v(x, x'))\partial u_0(x')/\partial x'. \qquad (6)$$

Here $T_v(x, x')$ is the *transit* time of a particle between x' and x starting at x' with velocity v'. Now v is the velocity at x of a particle starting with v' at x' in the total potential $-e\varphi + mu_0^2/4 \equiv \frac{1}{2}mv_\varphi^2(x)$ so that $v' = v(x, x') \equiv [v^2 + v_\varphi^2(x) - v_\varphi^2(x')]^{1/2}$. The total potential $-e\varphi_{\text{tot}} = \frac{1}{2}mv_\varphi^2(x)$ determining the orbits is repulsive and as shown in Fig. 1(b) the orbits separate into those which reflect from the potential and those which pass through. The choice of positive square root for v or v' in the expressions above explicitly limits us to electrons which pass through the potential. We assume that at large distances from the soliton the distribution is given—for example, a Maxwellian. The right-hand side is a spatially dependent diffusion term in velocity space which produces heating of the electrons as they stream through the soliton. The effect of the diffusion term on reflected particles can be handled similarly but will not be treated explicitly here. In steady state $(\partial F_L/\partial t = 0)$ we can formally solve (5) in the case v

>0 in the form

$$F_L(x,v) = f_M([v^2 + v_\varphi^2(x)]^{1/2}) + \int_{-\infty}^{x} dx'\, S(x', [v^2 + v_\varphi^2(x) - v_\varphi^2(x')]^{1/2}), \tag{7}$$

where $vS(x,v)$ is the right-hand side of Eq. (5) and $f_M(v^2) = f_0 \exp(-v^2/2ve^2)$ is the Maxwellian distribution at $x = -\infty$. In the region $v_\varphi(x) \neq 0$ the Maxwellian has v^2 replaced by $v^2 + v_\varphi^2(x)$ and the density is locally reduced by $\exp[-v_\varphi^2(x)/2v_e^2]$. We will look for steady-state solutions of Eq. (2) and make the static-ion approximation.[1,3] Poisson's equation and the assumption of quasineutrality then lead to the following relation between the low-frequency electrostatic potential and the ponderomotive potential: $v_\varphi^2 = T_e(T_e + T_i)^{-1} u_0^2/2$.

We also assume that $d\mathcal{E}_0(x)/dx$ is localized near $x = \pm l/2$, envisioning shapes of $\mathcal{E}_0(x)$ as shown in Fig. 1(a). The potential energy $\frac{1}{2}mv_\varphi^2(x)$ will then have a similar shape provided the jumps occur over regions several Debye lengths wide to preserve the quasineutrality assumption. Provided the other quantities in the integrand of (6) vary relatively slowly in x' we then use the approximation $du_0/dx \cong u_0[\delta(x + l/2) - \delta(x - l/2)]$ to perform the integrations over x and x'. If l_1 is the *width* of the step region of the soliton the conditions of validity of this approximation are that $x_H \approx u_0/\omega_0 \ll l_1 \ll v/\omega_0$ since v/ω_0 is approximately the spatial period of the oscillatory factor $\cos[\omega_0 T_v(x,x')]$. For $v > 0$ and $x < -l/2$ the distribution is Maxwellian as assumed at $x = -\infty$; i.e., $F_L(x,v) = f_M(v^2)$, $v > 0$ and $x < -l/2$. For $x \geq -l/2$ the distribution is altered by interaction with the soliton and we obtain the following equations:

$$F_L(-\tfrac{1}{2}l, v) = f_M(v^2 + \tfrac{1}{2}v_\varphi^2) + v_0^2\, \partial^2 F_L(-\tfrac{1}{2}l, v)/\partial v^2, \quad x = -\tfrac{1}{2}l; \tag{8a}$$

$$F_L(x,v) = f_M(v^2 + v_\varphi^2) + 4v_0^2\, \partial^2 F_L(-\tfrac{1}{2}l, v)/\partial v^2, \quad -\tfrac{1}{2}l < x < \tfrac{1}{2}l; \tag{8b}$$

$$F_L(\tfrac{1}{2}l, v) = f_M(v^2 + \tfrac{1}{2}v_\varphi^2) + 4v_0^2\{(\partial/\partial v)[1 - \cos(v_l/v)]\partial F_L(-\tfrac{1}{2}l, v)/\partial v\} + v_0^2\, \partial^2 F_L(\tfrac{1}{2}l, v)/\partial v^2, \quad x = \tfrac{1}{2}l; \tag{8c}$$

$$F_L(x,v) = f_M(v^2) + 4v_0^2\{(\partial/\partial v)[1 - 2\cos(v_l/v)]\partial F_L(-\tfrac{1}{2}l, v/\partial v\} + 4v_0^2\, \partial^2 F_L(\tfrac{1}{2}l, v)/\partial v^2, \quad x > \tfrac{1}{2}l; \tag{8d}$$

where $v_0 = u_0/4 = e\mathcal{E}_0/4m\omega_0$ and $v_l \equiv l\omega_0$. We have only to solve the second and fourth of these equations with appropriate boundary conditions. We have taken $v_\varphi^2(\pm\frac{1}{2}l) = \frac{1}{2}v_\varphi^2$, where v_φ^2 is the maximum value in $-l/2 < x < l/2$.

The solutions of the homogeneous equations are of the form $\exp(\pm v/v_0)$. One boundary condition is that $F(\pm l/2, v) \to 0$ as $v \to \infty$.

The second condition is obtained by examining the low-velocity behavior of Eq. (7) where $v < l_1\omega_0$. Here the integral term in (7) tends to vanish since the $\cos[\omega_0 T_v(x,x')]$ in $S(x',v)$ [see (6)] oscillates very rapidly. The particular solutions of Eqs. (8) which go continuously into the correct solution for small v can be shown to be that for which $\partial F/\partial v(\pm l/2, v) = 0$ at $v = 0$. Exactly similar results for large negative v are obtained by considering those trajectories originating at $x = +\infty$. The only change is to reverse the directional sense, e.g., $l/2 \to -l/2$.

These results are valid for $u_0 \ll l_1\omega_0 \ll v$. In a typical simulation result $l_1\omega_0 \sim 5v_e$ and $u_0 \sim v_e$. For low velocities $v \ll l_1\omega_0$ the heating effect vanishes as we have discussed and the distribution is just $f(x,v) = f_M(v^2 + v_\varphi^2(x))$. If $u_0 \lesssim v_e$ the bulk of the distribution function has this form and the electron density $n_L(x)$ is still given by $n_0(x)\exp[-m \times u_0^2(x)/T_e + T_i)]$ to terms of order $u_0/l_1\omega_0$. In Fig. 2 we plot $F_L(x,v)$ for $x > l/2$ for various values of v_0/v_e. Here we have $T_e \gg T_i$ so that $v_\varphi^2 = u_0^2/4$. An approximate but accurate formula for $F_L(x,v)$ for $x > l/2$ $(v > 0)$ which neglects some os-

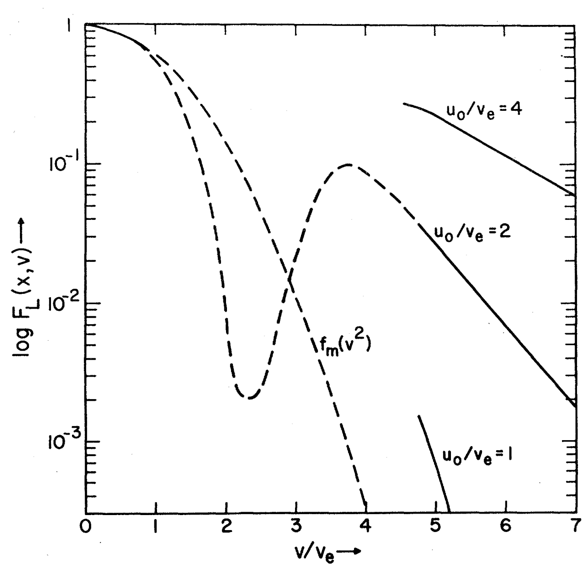

FIG. 2. Electron velocity distribution $F_L(x,v)$ for $x \gg l/2$, $v > 0$.

cillatory terms of order v/v_l when $v \ll v_l$ but which is valid also for $v \gg v_l$ is found for $u_0 \ll \omega_0 l_1 \ll v$:

$$F_L(x, v) = f_M(v^2)(9 + 4(v_e/v_0)^2\{[A(v)/v_e - v_e/v_0 - 2v_0/v_e]E(v_e/v_0 - v/v_e) + (v \to -v)\}) \\ = 4(2\pi)^{1/2}(v_e/v_0)^2 f_M(0)\exp[\tfrac{1}{2}(v_e/v_0)^2 - v/v_0]A(v)/v_e \text{ for } v \gg v_e^2/v_0, \tag{9}$$

where

$$E(x) = (\tfrac{1}{2}\pi)^{1/2}\exp(\tfrac{1}{2}x^2)\operatorname{erfc}(x/\sqrt{2}), \quad A(v) = v(1 - \cos v_l/v) - \operatorname{Si}(v_l/v) + \tfrac{1}{2}\pi v_l/v;$$

erfc is the complementary error function, and Si the sine integral function. This exponential dependence on v for large v is often seen in simulations.[2,6] The dashed intermediate-velocity portions of Fig. 2 are sensitive to the detailed shape of the soliton and have been treated only qualitatively. The condition of zero net current $\int_{-\infty}^{\infty} dv\, vF_L(x, v) = 0$ $(x \gg l/2)$ implies that $F_L(x, v)$ must cut below $f_M(v^2)$ in this region to compensate for the current in the high-velocity tail. This notch in the distribution appears in phase-space plots from computer simulations.[4]

The gain in electron kinetic energy is exactly balanced by loss of energy from the high-frequency electrostatic field. This is accounted for by the term on the right-hand side of (5). If we solve for F_H in terms of $\partial F_L/\partial v$ by using the Green's function solution it can be shown that the result is equivalent to adding the following term to the left-hand side of the NLSE:

$$i\int dx'\, Q(x, x', \{\mathcal{E}_0\})\,\partial\mathcal{E}_0(x')/\partial x', \tag{10}$$

where

$$Q(x, x', \{\mathcal{E}_0\}) = (4\pi e^2/m)\int_{-\infty,\ T_v(x,x')>0}^{\infty} dv\, v\sin[\omega_0 T_v(x, x')]\,\partial F_L(x', v)/\partial v \tag{11}$$

and $T_v(x, x')$ is, again, the transit time from x' to x starting with velocity v. The linear Landau damping is obtained in the limit $\mathcal{E}_0 \to 0$ where $T_v(x, x') = (x - x')/v$ and $F_L(v) = f_M(v^2)$. $Q(x, x', \{\mathcal{E}_0\})$ can be evaluated by use of $T_v(x, x') = (x - x')/v$ and the asymptotic result

$$F_L(x, v) = 2(2\pi)^{1/2}(v_e/v_0)\exp[\tfrac{1}{2}(v_e/v_0)^2 - v/v_0]f_M(0) \tag{12}$$

for $-\frac{1}{2}l < x' < \frac{1}{2}l$.

$Q(x, x', \{\mathcal{E}_0\})$ is a decaying, oscillatory function of $(\omega_0|x - x'|/v_0)^{1/2}$. If the range of $|x - x'|$ in the integration of Eq. (10), which is essentially l_1, the width of the peaks in $\partial\mathcal{E}_0/\partial x$, is large compared to this spatial periodicity then the oscillations of the integrand will wash out the integral leaving a small damping $\ll \omega_p$. However as the width of the soliton becomes comparable to v_0/ω_0 there are fewer oscillations of the integrand and a large damping results. Thus this damping stabilizes the soliton against spatial collapse.

The asymptotic behavior $F_L \to v\exp(-v/v_0)$ for $x > l/2$ and $v \gg v_0$ does not appear to be sensitive to the detailed shape of the soliton nor does the *qualitative* feature of phase-space streamers for $v \sim \omega_0 l_1$. The self-consistent determination of the soliton shape including Landau damping appears to require a detailed numerical solution of the modified NLSE.

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