## COMMENTS

# Comments on a Solution of a One-Dimensional Fermi-Gas Model

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A recent solution by Luther and Emery of a one-dimensional interacting Fermi-gas model contains a flaw in its final step which when corrected yields qualitatively different behavior. In particular spin-density wave and triplet superconducting pairing excitations are not divergent. The result is consistent with a gap in the uniform spin susceptibility and the gap can now be given a simple physical interpretation. The bulk of the Luther-Emery solution including exponents for the charge-density wave and singlet pairing remains correct.

Recently Luther and Emery (LE)[1] have produced a remarkable solution of a model for a one-dimensional interacting Fermi gas. The model is an extension of the Luttinger or Tomonaga model[2] to include spin as well as scattering from $+k_F$ to $-k_F$. The Hamiltonian is written as $H=H_S+H_L$ where $H_S$ is the usual Luttinger model,

$$H_S=v_F\sum_{k,s}k(a_{k,s}^\dagger a_{k,s}-b_{k,s}^\dagger b_{ks})+2L^{-1}\sum_k V\rho_1(k)\rho_2(-k), \tag{1}$$

where $a_{ks}$ ($b_{k,s}$) describes spin-$\frac{1}{2}$ fermions with momentum $k$ ($-k$); $\rho_1(k)$ and $\rho_2(k)$ are density operators,

$$\rho_1(k)=2^{-1/2}\sum_{ps}a_{p+k,s}^\dagger a_{ps}$$

and

$$\rho_2(k)=2^{-1/2}\sum_{ps}b_{p+k,s}^\dagger b_{ps}.$$

The large-momentum-transfer terms are described by

$$H_L=\sum_{ss'}\int dx\,\Psi_{1s}^\dagger(x)\Psi_{2s'}^\dagger(x)\Psi_{1,s'}(x)\Psi_{2,s}(x)[U_\parallel\delta_{s,s'}+U_\perp\delta_{s,-s}], \tag{2}$$

where $\Psi_{1s}(x)=L^{-1/2}\sum_k\exp(ikx)a_{k,s}$ and $\Psi_{2s}(x)=L^{-1/2}\sum_k\exp(ikx)b_{k,s}$. A special case of this model, $g_1\equiv U_\parallel=U_\perp$, has been investigated by Menyhárd and Sólyom[3] and recently extended by Fukuyama, Rice, Varma, and Halperin[4] using the renormalization-group method. For $g_1$ negative they find that the susceptibilities for charge-density wave and for singlet superconducting pairing diverge and that the uniform magnetic susceptibility goes to zero. LE solved the model for one particular value of $U_\parallel(2\pi v_F)^{-1}=-\frac{3}{5}$. The most striking feature of their solution is that they find an energy gap in the spin degrees of freedom and consequently the uniform magnetic susceptibility is exponentially activated. At the same time their low-temperature solution does not distinguish between spin singlet and triplet. They find that the susceptibility for spin- and charge-density waves behaves like $\omega^\mu$ for small $\omega$ and singlet and triplet superconductivity like $\omega^{\mu'}$. For $V(2\pi v_F)^{-1}>0$ they find $\mu<0$ and $\mu'>0$ indicating simultaneous charge- and spin-density wave instability, and $\mu'<0$, $\mu>0$ for $V(2\pi v_F)^{-1}<-\frac{3}{5}$ indicating simultaneous singlet and triplet superconducting instability. This contradicts the result of Menyhárd and Sólyom[3] which admittedly is not rigorous for $g_1<0$. It is the purpose of this Comment to point out an error in the last step of the LE argument which, when corrected, indicates that spin-density wave and triplet superconductivity responses are not divergent. The bulk of their remarkable solution, including the exponents $\mu$ and $\mu'$ for charge-density wave and singlet pairing, are still correct. Their result on the conductivity which is based on the charge-density-wave exponent alone also stands.

I first outline the steps in the LE solution to establish notations. Using the Mattis-Lieb[2] replacement of the kinetic energy term LE showed that $H$ can be decomposed into charge-density and spin-density components, i.e., $H=H_0+H_1$ where $[H_0,H_1]=0$ and

$$H_0=2\pi v_F L^{-1}\sum_k[\rho_1(k)\rho_1(-k)+\rho_2(-k)\rho_2(k)]+L^{-1}\sum_k(2V-U_\parallel)\rho_1(k)\rho_2(-k) \tag{3}$$

and

$$H_1 = 2\pi v_F L^{-1}\sum_k [\sigma_1(k)\sigma_1(-k) + \sigma_2(-k)\sigma_2(k)] - L^{-1}\sum_k U_\parallel \sigma_1(k)\sigma_2(-k) + U_\perp(2\pi\alpha)^{-2}\int dx\{\exp(2^{1/2}[\varphi_1(x)+\varphi_2(x)]) + \text{c.c.}\}, \tag{4}$$

where

$$\varphi_j(x) = 2\pi L^{-1}\sum_k k^{-1}\exp(-\tfrac{1}{2}\alpha|k| - ikx)\sigma_j(k), \tag{5}$$

and $\rho_1(k)$ and $\sigma_1(k)$ are the charge- and spin-density-wave operators defined by $2^{-1/2}\sum_p(a_{p+k\uparrow}{}^\dagger a_{p\uparrow} \pm a_{p+k\downarrow}{}^\dagger a_{p\downarrow})$, and $\rho_2$, $\sigma_2$ are similarly defined. In Eq. (4) the boson representation[5,6]

$$\Psi_{js}(x) \to \overline{\Psi}_{js}(x) = \lim_{\alpha\to 0}(2\pi\alpha)^{-1/2}\exp[\pm ik_F x + 2\pi L^{-1}\sum_k k^{-1}\exp(-\tfrac{1}{2}\alpha|k| - ikx)\rho_{js}(k)] \tag{6}$$

has been used to replace the four fermion operators in the $U_\perp$ term. In Eq. (6) the plus (minus) sign goes with $j=1$ $(j=2)$ and $\alpha^{-1}v_F$ is a cutoff parameter which LE interpret as a bandwidth. The next step is to perform a canonical transformation to diagonalize the part of $H_1$ excluding the $U_\perp$ term. The only effect on the $U_\perp$ term is to replace the factor in the exponent[7] by $2^{1/2}e^{\varphi}[\varphi_1(x)+\varphi_2(x)]$, where $\tanh 2\varphi = U_\parallel(2\pi v_F)^{-1}$. One then observes that for $2^{-1/2}e^{\varphi}=1$, which implies a particular value of $U_\parallel$, the $U_\perp$ term is precisely in the form of a boson representation of some fictitious spinless fermion field. Rewriting the kinetic energy term in fermion representation one finds

$$\tilde{H}_1 \equiv e^{iS}H_1e^{-iS} = v_F{}'\sum_k k(a_k{}^\dagger a_k - b_k{}^\dagger b_k) + U_\perp(2\pi\alpha)^{-1}\sum_k(a_k{}^\dagger b_{k-2k_F} + \text{H.c.}). \tag{7}$$

This is readily diagonalized resulting in an energy gap $\Delta = U_\perp(2\pi\alpha)^{-1}$.

Now that $H_1$ is diagonalized LE proceeded to calculate various response functions. Let us focus our attention on the spin-density-wave response $\chi(x,t)$ defined by

$$\chi(x,t) = \langle\Psi_{2\downarrow}{}^\dagger\Psi_{1\uparrow}(x,t)\Psi_{1\uparrow}{}^\dagger\Psi_{2\downarrow}(0,0)\rangle. \tag{8}$$

The response function is calculated using the boson representation

$$\overline{\Psi}_{1\uparrow}{}^\dagger\overline{\Psi}_{2\downarrow}(x,t) = (2\pi\alpha)^{-1}\exp(-2k_Fx)\exp\{-\sum_k A(x,k)2^{-1/2}[\rho_1(k,t)+\rho_2(k,t)]\} \times\exp\{-\sum_k A(x,k)2^{-1/2}[\sigma_1(k,t)-\sigma_2(k,t)]\}, \tag{9}$$

where $A(x,k) = 2\pi L^{-1}k^{-1}\exp(-\frac{1}{2}\alpha|k| - ikx)$. Since the $\sigma$ and $\rho$ degrees of freedom are completely independent, LE correctly pointed out that $\chi(x,t)$ can be factorized into two parts involving only $\rho$ or $\sigma$ to be calculated using $H_0$ and $H_1$, respectively. But then they claim that since $H_1$ has a gap in its spectrum at low temperatures only excitations in $H_0$ are important. Let us examine this claim by calculating the correlation function in $\sigma$ space explicitly. We first perform the canonical transform $e^{iS}$ which we saw had the effect of multiplying $\sigma_1+\sigma_2$ by $e^{\varphi}$. It is easy to see that the effect on $2^{-1/2}(\sigma_1-\sigma_2)$ is to replace it by $2^{-1/2}e^{-\varphi}(\sigma_1-\sigma_2)$. For the special choice of coupling constant the coefficient of $\sigma_1-\sigma_2$ is again unity and we can again use the boson representation. Considering only zero temperature we have

$$\tilde{\chi}(x,t) \equiv \langle 0|\exp[\sum_k A(\sigma_1-\sigma_2)]_{x,t}\exp[-\sum_k A(\sigma_1-\sigma_2)]_{0,0}|0\rangle_{\tilde{H}_1} = (2\pi\alpha)^2\langle 0|\Psi_2{}^\dagger(x,t)\Psi_1{}^\dagger(x,t)\Psi_1(0,0)\Psi_2(0,0)|0\rangle_{\tilde{H}_1}, \tag{10}$$

where $\Psi_1$ and $\Psi_2$ are the spatial representations of the operators $a$ and $b$ defined in Eq. (7) and $|0\rangle$ is the ground state of $\tilde{H}_1$. The point is that $\sigma_1-\sigma_2$ corresponds to a pair of destruction operators. A spectral analysis of Eq. (10) shows that the intermediate state must have a minimum energy of $2\Delta$ corresponding to creation of a pair of holes. To be more explicit we make use of the canonical transformation

$$\bar{a}_k = \cos\theta_k a_k - \sin\theta_k b_{k-2k_F}, \quad \bar{b}_{k-2k_F} = \cos\theta_k b_{k-2k_F} + \sin\theta_k a_k, \tag{11}$$

where $\tan 2\theta_k = U_\perp(2\pi\alpha)^{-1}(k-k_F)^{-1}$, to diagonalize $\tilde{H}_1$. Then $\tilde{\chi}(x,t)$ given in Eq. (10) can be computed to give

$$\tilde{\chi}(x,t)=(2\pi\alpha)^2\int\frac{dk}{2\pi}\frac{dq}{2\pi}\;\exp\{iqx_e i[E(k)+E(-k+q)]t\}\,[s_k c_{-k+q}(s_k c_{-k+q}+c_k s_{-k+q})],\tag{12}$$

where $E(k)=[U_\perp^2(2\pi\alpha)^{-2}+(k-k_F)^2]^{1/2}$, $s_k=\sin\theta_k$, and $c_k=\cos\theta_k$. Making the change of variable $\alpha k\to k'$, $\alpha q\to q'$, Eq. (12) is clearly of the form

$$\tilde{\chi}(x,t)=F(x/\alpha,v_F't/\alpha),\tag{13}$$

where $F(\bar{x},\bar{t})$ is independent of $\alpha$ and is a well-defined function which is oscillatory in $\bar{t}$. It is clear from Eq. (12) that $\tilde{\chi}$ has Fourier component only for $|\omega|>2\Delta$. $\tilde{\chi}(q,\omega)$ thus has a gap in its spectrum. The correlation function $\chi(q,\omega)$ is a convolution of $\tilde{\chi}(q,\omega)$ and $\chi_\rho(q,\omega)$ from the $\rho$ degrees of freedom. Details of $\chi(q,\omega)$ will depend on the behavior of $\chi_\rho(q,\omega)$ near $\omega=2\Delta$ which for $\Delta\approx\alpha^{-1}$ is not known reliably. However, it is clear that existence of a gap in $\tilde{\chi}(q,\omega)$ means that no divergence is possible in $\chi(q,\omega)$ for small $\omega$. This is to be contrasted with the charge-density-wave response function where $\sigma_1-\sigma_2$ in Eq. (9) is replaced by $\sigma_1+\sigma_2$. Then, in fact,

$$\langle 0|\exp[2^{-1/2}e^{\varphi}\textstyle\sum_k A(\sigma_1+\sigma_2)]|0\rangle_{H_1}\neq 0$$

and the low-lying states are completely described by the $\rho_1$ and $\rho_2$ degrees of freedom. An identical argument applies to the singlet and triplet superconducting pairing response.

Equation (13) clearly indicates a basic difficulty with the Luttinger-type model with $\delta$-function interaction, namely, that the problem as defined by Eqs. (1) and (2) contains no time scale. Hence any dimensionless function of space and time must appear with a cutoff $\alpha$. In the limit $\alpha\to 0$ the original problem becomes undefined. In keeping with LE's procedure we interpret $v_F\alpha^{-1}$ as a bandwidth. In the limit $\alpha\to 0$ not only are the gap and $\tilde{\chi}(x,t)$ undefined, but the correlation functions that LE calculated which are of the form $(\alpha\omega)^\mu$ are also either zero or infinite. Another way to illustrate this difficulty is to go to the limit of the Hubbard model where $U_\perp-U_\parallel=V=g$. The interaction is then simply

$$g\sum_{i>j}\delta(r_i-r_j).$$

If we have a parabolic band the strong-coupling limit of this model with attractive $g$ consists of singlet pairs of electrons.[8] The binding energy per pair can be worked out in terms of the ground-state energy of a particle bound by a $\delta$-function potential and is given by $g^2m\hbar^{-2}/4$. The energy gap is then cutoff independent. On the other hand if instead we have a linear spectrum that is cut off by a bandwidth $v_F\alpha^{-1}$ we can again solve for the binding energy of the singlet pair. This is essentially the problem solved by Cooper[9] and the binding energy per pair is

$$2\Delta'=v_F\alpha^{-1}\{\exp[(2g)^{-1}(2\pi v_F)]-1\}^{-1}.\tag{14}$$

It is interesting that in the limit of strong coupling $\Delta'$ reduces to the LE expression for the gap $\Delta$. Of course this type of argument does not constitute a proof that the original problem with a finite bandwidth is solved by the choice of a finite $\alpha$ in the end but it strongly suggests that this procedure is a reasonable one.

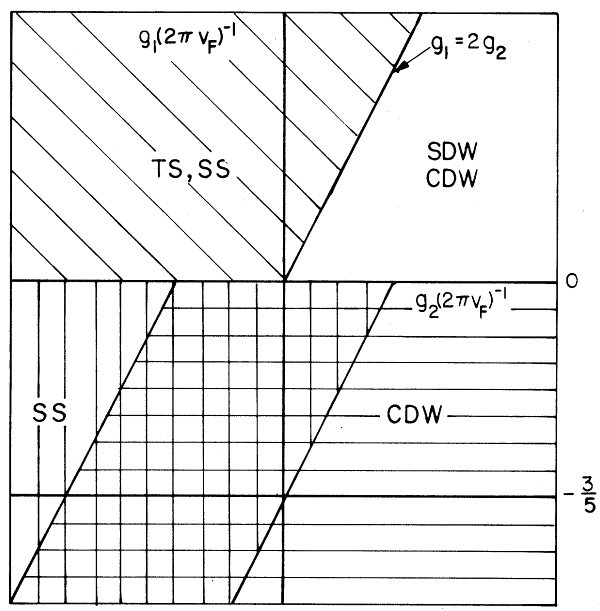


FIG. 1. Summary of our understanding of the zero-temperature interacting Fermi-gas problem where $g_1=U_\perp=U_\parallel$ and $g_2=V$. The behavior for $g_1>0$ is obtained using renormalization-group equations (Refs. 3 and 4). For $g_1>2g_2$ we have simultaneous triplet (TS) and singlet superconducting (SS) instability while for $g_1<2g_2$ we have charge- and spin-density wave (CDW and SDW). The line at $g_1(2\pi v_F)^{-1}=-\frac{3}{5}$ is where the LE solution is available. That solution is extended to the entire $g_1<0$ region by assuming that scaling laws are applicable. We have SS diverging on the extreme left and CDW on the right. The intermediate region is where both responses diverge. For $g_1<0$ SDW and TS responses are not divergent.

Our present understanding of the $g_1g_2$ problem is summarized in Fig. 1. The $g_1>0$ half-plane behavior is the renormalization-group result of Refs. 3 and 4. The uniform magnetic susceptibility is finite. The line at $g_1(2\pi v_F)^{-1}=-\frac{3}{5}$ corresponds to the LE solution. The rest of the $g_1<0$ plane is the expected behavior if scaling equations can be used to scale onto the LE solution. The uniform susceptibility is exponentially activated and all triplet excitations are nondivergent for $g_1<0$. The behavior is thus different from, but not completely orthogonal to, that suggested by Sólyom, which has simultaneous charge-density wave and singlet pairing instability for all $g_1<0$ and the triplet excitations behave like $\omega^\mu$ with $\mu>0$.

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[7]There appear to be misprints in LE so that they have $2^{-1/2}$ instead of $2^{1/2}$ in their Eq. (6). Their sign of $\varphi$ is also different so the final result remains unchanged.

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## Comments on "Is Bound Charm Found?"

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The inadequacy of the *Ansatz* proposed by De Rújula and Glashow for explaining the meson masses is discussed. The importance of fixing the spin-parity of the abnormal parity states $E(1420)$ and $D(1285)$ for the possible success of SU(4) is emphasized. We also point out some further consequences of the ideas of asymptotic freedom.

In a recent paper De Rújula and Glashow[1] propose the following *Ansatz* for the mass matrix

$$M_{ab,\,\alpha\beta}=\delta_{a\alpha,\,b\beta}(\mu_a+\mu_b+A)+\delta_{ab}\delta_{\alpha\beta}B, \tag{1}$$

where $A$ and $B$ depend on the spin, parity, etc., of the multiplet. By diagonalizing this matrix it is easy to see that

$$3B=\eta'+\eta-2K \tag{2}$$

and

$$(\eta'-\eta)^2/4=(K-\pi)^2+\tfrac{1}{4}(\eta'+\eta-2K)^2 - \tfrac{1}{3}(\eta'+\eta-2K)(K-\pi). \tag{3}$$

The last equation is poorly satisfied by pseudoscalar masses linear or quadratic. For comparison we write the SU(3) nonet mass formula

$$\mathcal{L}_{\text{mass}}=m_0(\mathrm{Tr}P)^2+m\,\mathrm{Tr}P^2 +\alpha\,\mathrm{Tr}P\Delta P+\beta\,\mathrm{Tr}P\,\mathrm{Tr}\Delta P, \tag{4}$$

where $P$ is the matrix of meson fields, $\Delta_{ij}=\Delta_{33}\delta_{i3}\delta_{j33}$. Equating (1) and (4) we find that De Rújula and Glashow's *Ansatz* implies $m_0=B$, $m=2\mu_1+A$, $\alpha\Delta_{33}=(\mu_3-\mu_1)$, $\beta=0$. To get a nonzero $\beta$ from Eq. (1) we must make $B$ depend on the quark index. If we use experimental masses in (4) we get $\beta\approx-\frac{1}{2}m_0$. In other words, $B$ for $\eta'$ ($\lambda\bar{\lambda}$ quarks) is half as much as for $\pi$ ($\mathcal{P}\bar{\mathcal{P}}$ quarks). This is in the right direction if the ideas of asymptotic freedom have any validity at all for lowest-lying hadrons.

*Vector mesons.*—The Okubo *Ansatz*[2] gives $\beta=m_0=0$, i.e., $B=0$. It is difficult to take the value $B=\frac{1}{2}(m_\omega-m_\rho)$, as given by Ref. 1, seriously since[3] (a) a large part of the $\rho$-$\omega$ mass difference can be electromagnetic in origin [$m_\rho=770\pm10$ MeV, $m_\omega=782\pm0.6$ MeV]; (b) $\Gamma(\omega\to2\pi)>10(e_2/4\pi)^2\Gamma(\rho\to2\pi)$ which suggests considerable electromagnetic mixing of $\rho$ and $\omega$; (c) mass splitting between $K^{*0}$ and $K^{*+}$ is substantial, $m^0$