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Lamb Shift in a Strong Coulomb Potential*

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Results are given of a calculation of the self-energy radiative level shift of order α of the $2S_{1/2}$ and $2P_{1/2}$ states in a strong Coulomb potential. The shift is evaluated numerically to all orders in $Z\alpha$ for Z in the range 10–110. An estimate is obtained for the effect of terms of high order in $Z\alpha$ on the Lamb shift in hydrogen. With this estimate taken into account, the theoretical value is $S = 1057.864(14)$ MHz.

Recent experiments with a variety of hydrogen-like ions have determined the values of the Lamb shift in these systems.[1] Comparison of these values to the values of the Lamb shift predicted by quantum electrodynamics is one of the fundamental tests of the theory. Furthermore, carrying out this comparison over a wide range of values of the nuclear charge Z is important in order to test whether the theory correctly predicts the Z dependence of the Lamb shift. For atomic hydrogen, new experimental techniques in measuring the Lamb shift give promise of increasing the precision of the measurements by an order of magnitude.[2] Similar precision in the theory requires knowledge of the contribution of terms of high order in $Z\alpha$. For high-Z atoms, a comparison can be made between the experimental and theoretical binding energies of the innermost electrons.[3] In this case the theoretical value of the radiative level shift in a Coulomb potential with nuclear charge Z provides a first approximation to the shift of the corresponding level in the neutral atom with the same Z. For the applications listed above, it is necessary to have accurate values predicted by quantum electrodynamics for the radiative shift of levels in a Coulomb potential for a wide range of Z.

In this Letter I report the results of a calculation of the self-energy contribution to the Lamb shift of electron levels in a strong Coulomb potential. The self-energy radiative level shift of order α of the $2S_{1/2}$ and $2P_{1/2}$ states, corresponding to the Feynman diagram in Fig. 1(a), is considered to all orders in $Z\alpha$. I have evaluated it numerically with no approximations by a slightly modified version of a method used previously to evaluate the $1S_{1/2}$-state self-energy.[4] The evaluation has been done for values of Z given by $Z = 10, 20, 30, \ldots, 110$. I have estimated the small-
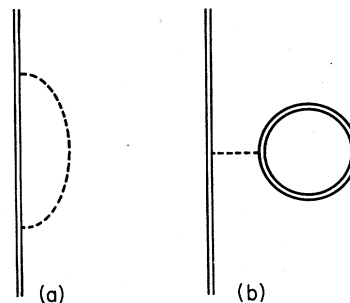

FIG. 1. Feynman diagrams for the radiative corrections of order α: (a) self-energy and (b) vacuum polarization.

TABLE I. Values for the function $F(Z\alpha)$.

Z	$F(Z\alpha)$	
	$2S_{1/2}$	$2P_{1/2}$
10	4.893(2)	−0.1145(4)
20	3.5063(4)	−0.0922(4)
30	2.8391(3)	−0.0641(4)
40	2.4550(3)	−0.0308(4)
50	2.2244(2)	0.0082(3)
60	2.0948(4)	0.0549(3)
70	2.0435(8)	0.1129(3)
80	2.065(2)	0.1884(3)
90	2.169(3)	0.2934(3)
100	2.387(3)	0.453(1)
110	2.798(3)	0.725(2)

FIG. 2. Calculated values of $G_{SE}(Z\alpha)$ for $Z = 10$ to 50 and the extrapolated value at $Z = 1$.

Z Lamb shift by extrapolating the calculated values with a procedure which takes into account the known behavior of the Lamb shift at small Z. My method of evaluating the energy shifts is based on the expansion of the bound-electron propagator in terms of the known Coulomb radial Green's functions. This expansion was used by Wichmann and Kroll in their study of vacuum polarization.[5] I employ the covariant regulator scheme to carry out the mass renormalization. Divergent terms and terms of order lower than $(Z\alpha)^4$ are isolated and treated analytically. Detailed results and modifications of the method necessary for the $n = 2$ states are described in a forthcoming paper.

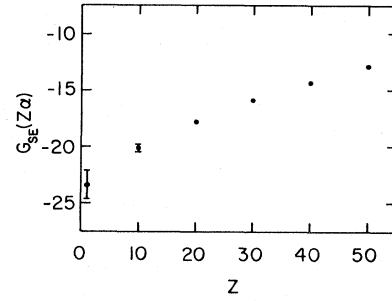Because of the approximate Z^4/n^3 scaling of the self-energy level shift ΔE, it is convenient to express the shift of each state in terms of a slowly varying function $F(Z\alpha)$ defined by

$$\Delta E = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{n^3} F(Z\alpha) mc^2, \tag{1}$$

where n is the principal quantum number of the state. Values I have obtained for $F(Z\alpha)$ are listed in Table I. The numbers in parentheses are estimates of the uncertainty associated with the numerical integration in the evaluation of $F(Z\alpha)$. Only the self-energy corresponding to Fig. 1(a) is included in $F(Z\alpha)$. Values of $F(Z\alpha)$ for intermediate values of Z can easily be obtained by ordinary polynomial interpolation. Erickson has obtained an approximation for the Z dependence of the Lamb shift which agrees qualitatively with my results.[6]

I obtain an estimate for the contribution of the terms of high order in $Z\alpha$ to the small-Z Lamb shift as follows: I isolate the known low-order terms of the self-energy (SE) contribution to the Lamb shift[7] $\mathcal{S}_{SE} = \Delta E(2S_{1/2}) - \Delta E(2P_{1/2})$ by writing

$$\mathcal{S}_{SE} = \frac{\alpha}{\pi} \frac{(Z\alpha)^4}{6} mc^2 \left[\ln(Z\alpha)^{-2} - \ln\frac{K_0(2,0)}{K_0(2,1)} + \tfrac{11}{24} + \tfrac{1}{2} + 3\pi(1 + \tfrac{11}{128} - \tfrac{1}{2}\ln 2)(Z\alpha) - \tfrac{3}{4}(Z\alpha)^2 \ln^2(Z\alpha)^{-2} \right.$$
$$\left. + (\tfrac{299}{240} + 4\ln 2)(Z\alpha)^2 \ln(Z\alpha)^{-2} + (Z\alpha)^2 G_{SE}(Z\alpha) \right]. \tag{2}$$

As a consequence of this definition, the function $G_{SE}(Z\alpha)$ approaches a constant as $Z\alpha \to 0$. We expect that the small-$Z\alpha$ behavior of $G_{SE}(Z\alpha)$ has the form

$$G_{SE}(Z\alpha) = a + b(Z\alpha)\ln(Z\alpha)^{-2} + c(Z\alpha) + \ldots, \tag{3}$$

where the omitted terms are higher order in $Z\alpha$. This behavior is suggested by the form of the high-order terms of the vacuum polarization. Fitting the function on the right-hand side of (3) to the values of $G_{SE}(Z\alpha)$ corresponding to my calculated values of the self-energy at $Z = 10$, 20, and 30 yields a value of $G_{SE}(\alpha) = -23.4 \pm 1.2$ for hydrogen. The upper and lower limits for $G_{SE}(\alpha)$ are obtained by similar extrapolations with the value of $G_{SE}(Z\alpha)$ at $Z = 10$ replaced by its upper and lower limits corresponding to the uncertainty listed in Table I. Although this procedure does not give rigorous limits to the error in $G_{SE}(\alpha)$, I feel that they are valid limits to the uncertainty. This view is supported by the fact that extrapolations both from the calculated points at $Z = 20$, 30, and 40 and from the calculated points at $Z = 30$, 40, and 50 yield a value for $G_{SE}(\alpha)$ within the stated limits. In addition, extrapolation from the

calculated points at $Z = 10$, 20, and 30 with an ordinary second degree polynomial in $Z\alpha$ yields a value within the limits. Figure 2 shows the calculated values of $G_{SE}(Z\alpha)$ from $Z = 10$ to 50. The error bar at $Z = 10$ corresponds to the uncertainty in $F(Z\alpha)$. The point at $Z = 1$ is the extrapolated value $G_{SE}(\alpha)$.

To order α, the remaining radiative correction to be included is the vacuum polarization corresponding to the Feynman diagram in Fig. 1(b). Wichmann and Kroll have considered this correction in detail.[5] They have shown that for small Z the dominant contribution is given by the Uehling potential which is the part of the vacuum polarization linear in the external potential. In particular, they found that the part which is third order in the external potential contributes only 308 Hz to the $2S_{1/2}$-state level shift in hydrogen. (Even powers in the external potential give no contribution as a consequence of Furry's theorem.) We express the total vacuum-polarization (VP) contribution of order α to the Lamb shift as

$$\mathcal{S}_{VP} = \pi^{-1}\alpha \tfrac{1}{8}(Z\alpha)^4 mc^2 \left[-\tfrac{1}{5} + \tfrac{5}{64}\pi(Z\alpha) - \tfrac{1}{10}(Z\alpha)^2 \ln(Z\alpha)^{-2} + (Z\alpha)^2 G_{VP}(Z\alpha) \right]. \tag{4}$$

The known low-order terms displayed in (4) are obtained from the Uehling potential.[7] In view of the Wichmann-Kroll result, the function $G_{VP}(Z\alpha)$ is well approximated for small Z by the part $G_U(Z\alpha)$ which arises from the Uehling potential. I have calculated the small-$Z\alpha$ behavior of $G_U(Z\alpha)$ and obtain

$$G_{VP}(Z\alpha) \approx G_U(Z\alpha) = -\tfrac{1199}{2100} + \tfrac{5}{128}\pi(Z\alpha)\ln(Z\alpha)^{-2} + 0.5(Z\alpha) + \dots . \tag{5}$$

The sum of the self-energy and vacuum-polarization contributions $G(Z\alpha) = G_{SE}(Z\alpha) + G_{VP}(Z\alpha)$ for hydrogen is $G(\alpha) = -24.0 \pm 1.2$ and gives a shift of $-0.173(9)$ MHz. The value corresponding to this correction according to the compilation of Lautrup, Peterman, and de Rafael is $-0.126(5)$ MHz.[7] That value is based on a calculation of the high-order binding correction by Erickson.[6] I do not know the source of the discrepancy between that value and my value. My result is consistent with the earlier estimate of Erickson and Yennie[8]: $G(0) = -19.08 \pm 5$. Combining my value for the high-order binding correction with the values for the other contribution listed by Lautrup, Peterman, and de Rafael, I obtain the theoretical Lamb shift value of $\mathcal{S} = 1057.864(14)$ MHz in hydrogen. The effect of the high-order binding terms is quite important in comparison to the accuracy of the recent preliminary experimental result,[9] $\mathcal{S} = 1057.893(20)$ MHz.

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