tional technique of this paper to the Gaussian model, using the same recursion method as used by Bell and Wilson.[10] For the Gaussian model, the present method gives a $\vec{K}^*$ which agrees exactly with the critical coupling strengths and gives precisely the right critical indices for all values of dimensionality. Further study will be required to establish why the errors of this procedure are so unexpectedly small.

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[4]The symmetries actually considered in this paper are translational invariance plus the point group of the square or cube.

[5]R. P. Feynman, *Statistical Mechanics: A Set of Lectures. Richard P. Feynman*, edited by D. Pines (Benjamin, Reading, Mass., 1972), pp. 66–71.

[6]This equation and others assume that all the variables commute. A generalization to noncommuting cases is easily obtained (see Ref. 5).

[7]Proof that $\mathcal{H}^L(\mu) \leq \mathcal{H}'(\mu)$: Let $\mathcal{H}^L(\mu)$ obey the lattice symmetries and be defined by Eq. (7), in which $V(\mu, \sigma)$ is odd under these symmetries. Then the standard methods (Ref. 5) suffice to prove that

$$\mathrm{Tr}_\mu X(\mu) e^{-\mathcal{H}^L(\mu)} \geq \mathrm{Tr}_\mu X(\mu) e^{-\mathcal{H}'(\mu)}$$

for any positive semidefinite $X(\mu)$ which obeys the lattice symmetries. Choose $X(\mu)$ to be projection operator onto all configurations of the μ lattice which are equivalent to μ under the symmetry operations. The desired result then follows.

[8]Such motion was actually used by Kadanoff and Houghton (Ref. 3) to get rid of some "hard" terms.

[9]K. Wilson, private communication; Th. Neimeijer and J. M. J. van Leeuwen, Phys. Rev. Lett. $\underline{31}$, 1412 (1973), and Physica (Utrecht) $\underline{71}$, 1974; M. Nauenberg and B. Nienhuis, Ref. 1; Kadanoff and Houghton, Ref. 3.

[10]T. L. Bell and K. G. Wilson, to be published.

Stability of the Critical Surface in Irradiated Plasma*†

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The linear and nonlinear evolution of an instability of one-dimensional filaments in plasma is investigated. The relevance of these results to two-dimensional breakup of the critical surface in laser-plasma interactions is pointed out.

The possibility of obtaining self-trapped solutions for radiation intensity in media which have a nonlinear index of refraction is well established.[1,2] The question of the stability of these states bears directly on their possible experimental observation. For the case of filamentary equilibria in plasma, it has been shown theoretically that such states are unstable to kink, or bending, perturbations.[3]

We derive here the properties of another, faster growing, instability of such filamentary structures in plasma. This instability is of the sausage, or necking, type. We also demonstrate by numerical solution of the nonlinear coupled electromagnetic and plasma-fluid equations that this instability may result in the destruction of planar symmetry of the critical surface in an expanding irradiated plasma. Such a two-dimensional breakup of the critical surface can modify the collective absorption mechanisms which occur there and which play a central role in many laser-fusion schemes. We note that the existence of this instability in cubically nonlinear media has been demonstrated by Zakharov[4] and Zakharov and Rubenchik.[5] These calculations do not allow a determination of the growth rate, however, because either the inertia of the nonlinearity[4,5] or the parallel dispersion of the trapped wave[5] is neglected. Both effects are important in the case of electromagnetic radiation trapped in plasma.

The equations we solve are the one-fluid plasma equations

$$\frac{\partial n}{\partial t} + \nabla \cdot n\vec{v} = 0, \tag{1a}$$

$$nM\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = \frac{-q^2 n}{2m\omega_0^2}\nabla\langle E^2\rangle - T\nabla n, \tag{1b}$$

for the density n and velocity $\vec{v}$, together with the electromagnetic wave equation

$$\partial^2 E/\partial t^2 - c^2\nabla^2 E = -(4\pi e^2/m)nE. \tag{2}$$

Here $E = \vec{E}\cdot\hat{z} = |\vec{E}|$ and we assume everywhere that $\partial/\partial z \equiv 0$. The ponderomotive force term, $\nabla\langle E^2\rangle$, describes the slowly varying momentum transfer between radiation and plasma. The brackets indicate an average over the short electromagnetic wave period $2\pi/\omega_0$.

We first analyze the stability of a static, one-dimensional, plane-stratified, isolated, filamentary equilibrium which varies with position in the x direction only. With these assumptions, Eq. (1b) can be solved for $n(E^2)$ and the result substituted into Eq. (2) to yield

$$L_0 E_0 \equiv \left\{\frac{d^2}{dx^2} + \frac{\omega_0^2}{c^2}\left[1 - n_0\exp(-E_0^2)\right]\right\}E_0 = 0, \tag{3}$$

where the transformations $4\pi e^2 n/m\omega_0^2 \to n$ and $eE/\omega_0(2Tm)^{1/2} \to E$ to dimensionless variables have been made, n_0 has been introduced to designate $n(E^2=0)$, and the electric field has been assumed to be of the form $E(x,t) = \sqrt{2}\,E_0(x)\cos(\omega_0 t)$. Additionally, the operator L_0 has been defined as indicated by the outer braces. Equation (3) has isolated filament solutions.[1,2]

In linear order, we assume variation with y and t of the form $\exp(iky - i\Omega t)$ and seek the eigenfrequency $\Omega(k^2)$. We adopt the following ordering for k and Ω:

$$kL < 1, \tag{4a}$$

$$c_s^2/L^2 < \Omega^2 < k^2c^2. \tag{4b}$$

Here L is the scale length for variation of the equilibrium. Use of inequality (4a) and the first inequality in (4b) in the linearized form of Eq. (1) facilitates the immediate solution for n which is given by

$$n = -(c_s/\Omega)^2(d/dx)n_0(d/dx)E_0F^+.$$

Here the convenient notation $F^\pm$ has been introduced through the assumed form for E as follows:

$$E(x,y,t) = 2^{-1/2}[E^+(x)\exp(i\omega_0 t) + E^-(x)\exp(-i\omega_0 t)]\exp(iky - i\Omega t),$$

and $F^\pm \equiv E^+ \pm E^-$. The result for n is then substituted into the following linearized equations for $F^\pm$:

$$L_0F^+ + \left(\frac{\Omega^2}{c^2} - k^2\right)F^+ - 2\frac{\Omega}{c^2}\omega_0 F^- = 2\frac{\omega_0^2}{c^2}nE_0, \tag{5a}$$

$$L_0F^- + \left(\frac{\Omega^2}{c^2} - k^2\right)F^- - 2\frac{\Omega}{c^2}\omega_0 F^+ = 0. \tag{5b}$$

The dispersion relation is readily obtained from these equations by exploiting the relative dominance of L_0 with the result

$$\frac{\Omega^2}{\omega_0^2} = \frac{1}{8}\left\{\left(\frac{kc}{\omega_0}\right)^4 \pm \left[\left(\frac{kc}{\omega_0}\right)^8 + 32\left(\frac{c_s c\lambda k}{\omega_0^2}\right)^2\right]^{1/2}\right\}, \tag{6}$$

where λ^2 is defined by

$$\lambda^2 \equiv \frac{\int dx (dE_0^2/dx)^2\exp(-E_0^2)}{\int dx E_0^2}.$$

If the inequality $kc/\omega_0 < (c_s\lambda/\omega_0)^{1/2}$ is also satisfied, then the dispersion relation becomes more simply $\Omega^2 = \pm gk$, with an effective acceleration $g = c_s c\lambda/2$.

We have developed a computer code FLUID which advances the full nonlinear Eqs. (1) and (2) with the addition of a term $\mu\nabla^2\vec{v}$ to the right-hand side of Eq. (1b). This term is added to model ion dissipative effects, but is chosen small enough to have a negligible effect on the results presented below. A leap-frog algorithm is used to advance the equations. The boundary conditions are periodic in one direction (y), and transmitting for the radiation but reflecting for the plasma in the other (x).

The instability was studied under the ideal conditions discussed above. The parameters were as follows: initial peak field amplitude $E_0 = 1.0$, plasma periodicity length $L_y = 10\lambda_0$, thermal velocity $v_{te}/c = 0.1$, and electron-to-ion mass ratio $m/M = 0.01$. The filamentary equilibrium was constructed from a numerical solution of Eq. (3), where the parameter choice $n_0 = 1.58$, $E(0) = 1$, $dE(0)/dx = 0$, yields an isolated filament. A small level of random noise was initially superimposed on this otherwise static equilibrium.

A contour plot of the initial density profile is shown in Fig. 1(a). A plot of the spatial Fourier analysis of the density versus time, $|\int dxdy$

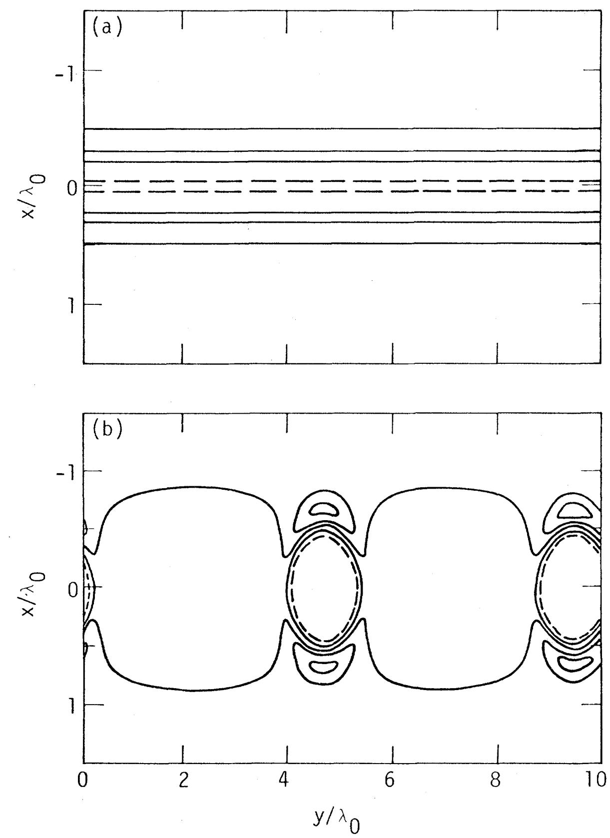

FIG. 1. Density contours at two stages of the evolution of the sausage instability. Parameters: maximum $E_0=1.0$, mass ratio $m/M=0.01$, periodicity length $L_y=10\lambda_0$, and electron thermal velocity $v_{te}/c=0.1$. (a) Contours at $t=0$. Density $n=0.6n_{cr}$ on the dashed contour. (b) Contours after saturation at $\omega_0 t=942$. Here $n=0.3n_{cr}$ on the dashed contour, and the small elliptical contours are near maxima. In both cases, $\Delta n=0.3n_{cr}$ between contours.

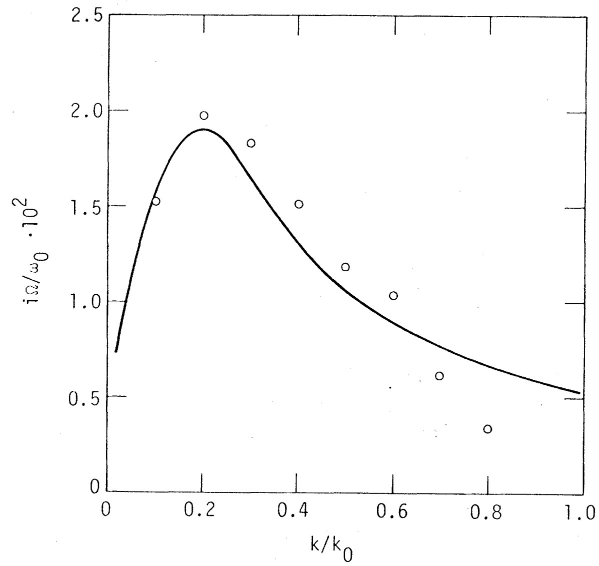

FIG. 2. Dots are observed values of instability growth rate $i\Omega$ versus y wave vector k for the simulation described in Fig. 1. The solid line is an evaluation of the dispersion relation, Eq. (6), for these parameters.

$\times n(x,y,t)\exp(iky)|$, shows exponential growth of the first eight modes for times $\omega_0 t \leqslant 780$. The observed growth rates are shown as points in Fig. 2, where the dispersion relation, Eq. (6), evaluated for these parameters, is plotted as a solid line. The agreement is accurate to within 20% for their first six modes, but the theory evidently breaks down as $kc/\omega_0=k/k_0\to 1$, as is to be expected. The linear phase is followed by one in which energy is transferred to higher mode numbers from the linearly fastest growing modes, and finally by saturation. Figure 1(b) shows a contour plot of density at saturation. The filament has evolved into a series of almost circular cavities with radiation trapped inside. Similar effects have been observed in investigations of Langmuir wave filaments in electrostatic plasma in which the coupled nonlinear Schrödinger and ion-sound-wave equations have been solved numerically.[6] That the electromagnetic and electrostatic results are comparable is not surprising in view of the similar structure of the two sets of equations.

The results of this model problem can be relevant to the more realistic radiation-plasma interaction problem in which radiation is incident on an inhomogeneous plasma. To illustrate this within the FLUID model, radiation was emitted normally from the left (x) boundary onto a plasma with an initial density profile which varied linearly with position from $n=0.3$ to $n=1.5$ in a distance $\Delta x=7\lambda_0$. The peak electric field amplitude was $E_0=0.5$ and the periodicity length $L_y=7\lambda_0$. The thermal velocity and mass ratio were unchanged from above. Upon reflection from the critical surface, the radiation sets up a standing wave pattern. The plasma response to the resulting spatial variation of the pondermotive force is to form periodic density depressions as shown in the surface plot of Fig. 3(a). These depressions are similar to the filaments discussed above,[7] and undergo the same two-dimensional collapse observed in the model problem. The resulting nonlinear state is shown in Fig. 3(b) at time $\omega_0 t=1130$. The maximum density on the

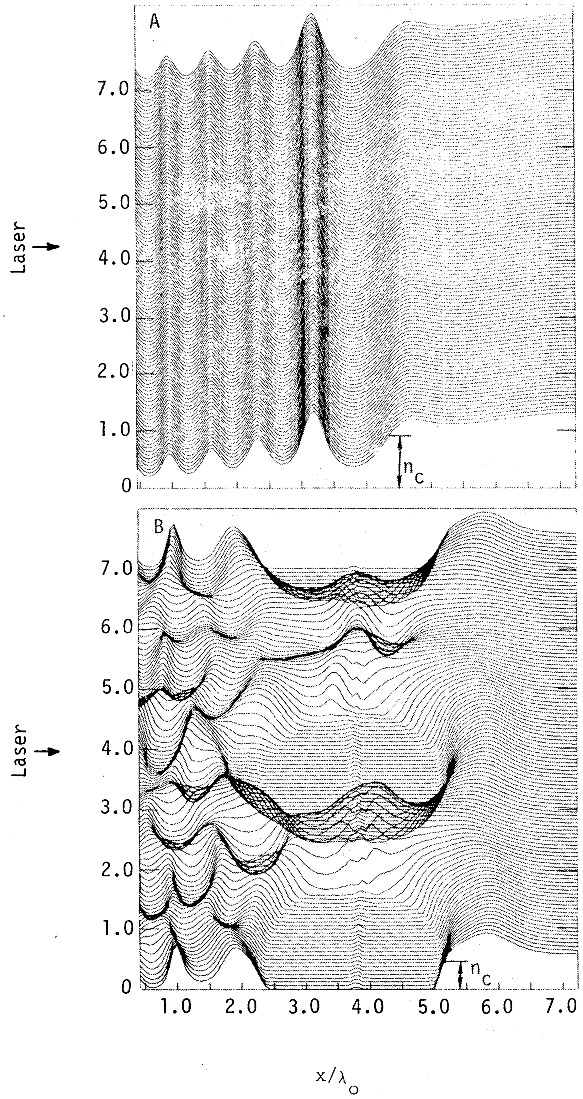

FIG. 3. Surface curves of the density of plasma which is irradiated normally from the left with radiation of amplitude $E_0=0.5$. The one-dimensional depressions of (a) at $\omega_0 t=314$ are drastically altered by the sausage instability to those shown in (b) at $\omega_0 t=1130$. Parameters: incident amplitude $E_0=0.5$, initial density variation linear from $n=0.3$ to $n=1.5$ in a distance $\Delta x=7\lambda_0$, periodicity length $L_y=7\lambda_0$, thermal velocity $v_{te}/c=0.1$, and mass ratio $m/M=0.01$. Arrows indicate the critical density level.

side of these cavities exceeds the critical density, again partially trapping radiation in them. At later times, because of plasma expansion, the structure elongates in x. Such tubular density cavities with boundary densities near or exceeding the critical density can lead to greater absorption than is predicted on the basis of results[8] for a planar critical surface.

We have verified that filament formation is an intrinsic part of this two-dimensional evolution by repeating the computation with all parameters identical except for the initial density profile. This varied discontinuously from $n=0.1$ to $n=1.5$, so that a trough would not form. The radiation-plasma interface remained planar in this case.

The instability is observed to occur for non-normally incident radiation. It is observed to occur independently in each of several troughs when the initial density gradient is shallower; that is, a series of "bubbles" forms in each of several troughs with no apparent relationship in their y positions to the y positions of those in other troughs. These effects have also been shown to occur for plasma which has an initial density everywhere slightly less than critical. We are continuing investigations along these lines and are also studying the effects of absorption at the critical surface on these results.

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