the direction of spin alignment in the A_1 phase is opposite to that which is predicted using weak-coupling theory. The magnitude of the linear splitting calculated using spin-fluctuation theory with $1 - IN(\epsilon_F) = \frac{1}{20}$, $\omega_{sf} = \epsilon_F/20$, and $N(\epsilon_F)V \cong 4.0$ (and the RPA approximation for μ^{eff}) is about 4 times larger than that measured experimentally; in weak-coupling theory, the calculated splitting differs from the experimental one by a factor of more than 25. While there are, at present, no measurements which can verify this prediction, it should shortly be possible[11] to ascertain the direction of the spin pairs in the A_1 state.

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[8]The validity of the static approximation for paramagnon-induced singlet-state pairing (when vertex corrections are not included and there are no "feedback effects") has been discussed in detail by N. F. Berk and J. R. Schrieffer [see, for example, Phys. Rev. Lett. 17, 433 (1966)] who find that $Z \sim 1 - N(\epsilon_F) I \ln[1 - N(\epsilon_F) I]$ which is *not* a small correction. However, because of additional complications, dynamical effects have yet to be included in a self-consistent treatment of superfluid ^{3}He.

[9]Higher order (in I) terms lead to unphysical expressions for μ^{eff}. It is believed that this behavior arises because of the inapplicability of the static approximation and the neglect of vertex corrections. While more elaborate calculations may be required to incorporate these effects, they are outside the scope of the paramagnon theories which have been previously discussed and which it is intended to explore here.

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Variational Principles and Approximate Renormalization Group Calculations*

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Approximate recursion relations which give upper and lower bounds on the free energy are described. "Optimal" calculations of the free energy can then be obtained by treating parameters within the renormalization equations variationally. As an example, simple lower-bound relations are defined for the two- and three-dimensional Ising models. At the fixed point, a parameter is set variationally, and then critical indices are calculated.

Several recent papers[1] have followed up on the pioneering work of Wilson[2] in using the renormalization group to perform approximate calculations of the free energy. In general, the problem is to calculate a free energy $F_N(\vec{K})$ for a system with N degrees of freedom and a set of coupling constants $\vec{K}$, starting from a Hamiltonian $\mathcal{H}_{\vec{K}}(\sigma)$, in which the σ represents some set of statistical variables.[3] A renormalization transformation $\vec{K}' = R(\vec{K})$ defines a new set of coupling parameters as a function of the old parameters. The $\vec{K}'$ defines couplings in a system with fewer degrees of freedom, $N' < N$, described by a new set of variables which we call μ. The special property of this transformation is that it leaves the total free energy invariant, i.e.,

$$F_N(\vec{K}) = F_{N'}(R(\vec{K})). \tag{1}$$

To realize such a transformation, define $F_N(\vec{K}) = -\ln \mathrm{Tr}_\sigma \exp[-\mathcal{H}_{\vec{K}}(\sigma)]$ and construct a new Hamiltonian as

$$\mathcal{H}'(\mu) = -\ln \mathrm{Tr}_\sigma \exp[S(\mu, \sigma) - \mathcal{H}_{\vec{K}}(\sigma)]. \tag{2}$$

If $S(\mu,\sigma)$ obeys the restriction $\mathrm{Tr}_\mu \exp S(\mu,\sigma)=1$, then $\mathcal{H}'$ and $\mathcal{H}_{\vec{K}}$ have exactly the same free energy. If the set of parameters $\vec{K}$ is sufficiently inclusive so that all conceivable Hamiltonians which obey a given symmetry[4] can be represented as $\mathcal{H}_{\vec{K}}(\sigma)$, and if in addition $S(\mu,\sigma)$ obeys this symmetry, then $\mathcal{H}'(\mu)$ must also be expressible as $\mathcal{H}_{\vec{K}'}(\mu)$ for some value of $\vec{K}'$. In this way, one realizes the renormalization group mappings.

A most important part of the Hamiltonian $\mathcal{H}_{\vec{K}}(\sigma)$ is a constant term, which I define to be $-NK_0$. In the evaluation of the free energy, one visualizes an infinite number of renormalizations

$$\vec{K}^\alpha = R(\vec{K}^{\alpha-1}), \quad \vec{K}^0 = \vec{K}. \tag{3}$$

After many renormalizations, the trivial term ${K_0}^\alpha$ will grow in magnitude, and, dominating all else, will provide an evaluation of the free energy as

$$F_N(\vec{K}) = -\lim_{\alpha\to\infty} N_\alpha {K_0}^\alpha. \tag{4}$$

Thus, if R is known, F can be computed.[1,2]

But R is not known. Instead, I seek a prescription for constructing upper and lower "bounding" approximations to R denoted as R^U and R^L, which respectively have the properties

$$F_{N'}(R^U(\vec{K})) \geq F_{N'}(R(\vec{K})) \geq F_{N'}(R^L(\vec{K})). \tag{5}$$

Note that if one uses upper (lower) boundary recursion relations in the sequence of calculations described by Eqs. (3), the resulting limit (4) will be an upper (lower) bound to the true free energy. Furthermore, any parameters in R^U or R^L (including parameters in S) can be varied so as to obtain optimum bounds on $F_N(\vec{K})$.

The standard variational principles in statistical mechanics[5] define appropriate R^U and R^L. Define a variational function $H_0(\mu,\sigma)$ which includes the basic symmetries and obeys $\mathrm{Tr}_\sigma \times \exp H_0(\mu,\sigma)=1$. Then notice that[6]

$$\mathcal{H}^U(\mu) = -\mathrm{Tr}_\sigma \exp[H_0(\mu,\sigma)] \times [S(\mu,\sigma) - \mathcal{H}_{\vec{K}}(\sigma) - \mathcal{H}_0(\mu,\sigma)] \tag{6}$$

must equal or exceed $\mathcal{H}'(\mu)$ for all values of μ —if $\vec{K}$ and S are held fixed. When $\mathcal{H}^U$ is parametrized in terms of $\vec{K}'$, Eq. (6) generates a recursion $\vec{K}' = R^U(\vec{K})$, which certainly obeys Eq. (5). Similarly define

$$\mathcal{H}^L(\mu) = -\ln \mathrm{Tr}_\sigma \times \exp[S(\mu,\sigma) - \mathcal{H}_{\vec{K}}(\sigma) + V(\mu,\sigma)]. \tag{7}$$

If we demand that $V(\mu,\sigma)$ is a sum of terms which are each odd under a lattice symmetry operation and that $\mathcal{H}^L$ obeys all the symmetries, then $\mathcal{H}^L(\mu) \leq \mathcal{H}'(\mu)$ for all values of μ. (This statement is easily proved.[7]) We can then once more parametrize in terms of $\vec{K}'$ and say that Eq. (7) generates a recursion relation $\vec{K}' = R^L(\vec{K})$ which obeys Eq. (5).

It is relatively easy to see how to derive upper bounds via Eq. (6). The transforms used in Kadanoff and Houghton[3] can easily be modified to fit into the form (6). It is harder to see that the form (7) is also usable. To write an odd representation of V, take $V(\mu,\sigma) = V_1(\sigma) + V_2(\mu)$. Define a set of functions $s_{i,j}(\sigma)$ in which the index i defines different kinds of terms—e.g., nearest-neighbor products or products of four spins in a square. Then j is essentially a position variable which distinguishes among different—but equivalent—terms. Now a suitable V_1 can be written as

$$V_1(\sigma) + \sum_{i,j} v_1(i,j) s_{i,j}(\sigma), \tag{8}$$

with $\sum_j v_1(i,j)=0$. Also, V_2 can be written in exactly the same form.

Notice that in Eq. (7) $V_1(\sigma)$ is added to $\mathcal{H}_{\vec{K}}(\sigma)$. *Then we may say that V_1 changes $\mathcal{H}_{\vec{K}}$ by moving*[8] *equivalent interaction bonds to different portions of the lattice.* A suitable choice of V_1 will break the bonds which connect spins in different regions of the lattice. Then the entire sum in Eq. (7) factorizes into a product of independent terms which each can be summed exactly.

Once the sum is calculated, in general $\mathcal{H}^L(\mu)$ will not obey the lattice symmetries. But since $V_2(\mu)$ is antisymmetric, a suitable choice of V_2 can always be found which will force $\mathcal{H}^L(\mu)$ to obey the symmetries. This $\mathcal{H}^L(\mu)$ is then used to calculate the new couplings $\vec{K}' = R^L(\vec{K})$.

For illustrative purposes, in this paper I carry through a simple example of this calculation on the square lattice as shown in Fig. 1. Within each square, I can define spin operators, s_0, s_1, s_{nn}, s_{nnn}, s_3, and s_4. Here s_0 is 1; s_1, s_3, and s_4 are respectively the sums of all possible products of 1, 3, and 4 spins in the square; while s_{nn} and s_{nnn} are respectively sums of products of all nearest-neighbor and next-nearest-neighbor pairs of spins. I write

$$\mathcal{H}_{\vec{K}} = -\sum_{\text{squares}} \sum_i K_i s_i. \tag{9}$$

Every fourth square contains a new spin $-\mu$. I call these red squares. I choose (see Kadanoff and Houghton[3]) an S which depends upon a varia-

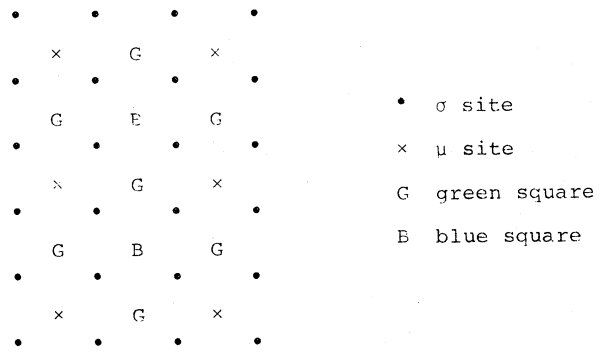

FIG. 1. Recursion setup on the $d=2$ lattice. The "red squares" are those with μ sites at their centers.

tional parameter, p:

$$S(\mu,\sigma)=\sum_{\substack{\text{red}\\ \text{squares}}}(\mu p s_1-\sum_i u_i s_i). \tag{10}$$

Here, the u's are functions of p chosen so that $\sum_i u_i s_i=\ln(2\cosh p s_1)$. Then, these terms in S insure that the sum over μ of e^S is exactly 1 so that the transformation preserves the value of F. Moreover, when p is well chosen, these terms will tend to subtract away some of the strongest terms in $\mathcal{H}_{\vec{K}}$, leaving a weaker and more tractable interaction.

However, some interactions must still be moved in order to make the sum in Eq. (7) tractable. To eliminate all the interaction terms in the red and green squares, choose

$$v_1(i,j)=\begin{cases}-K_i+u_i \text{ in red squares},\\ -K_i \text{ in the green},\\ 3K_i-u_i \text{ in the blue}.\end{cases} \tag{11}$$

Since *all* interactions now lie in the blue squares, the sum (7) can be split into a product of sums over blue squares. Each such sum contains only sixteen terms and can easily be calculated on the computer. These partial sums produce new couplings of up to four spins of exactly the form (9). [Since the resulting $\mathcal{H}^L(\mu)$ obeys the lattice symmetries, $V_2(\mu)$ is zero.] Therefore, by calculating some rather simple sums, I have found a lower-bound recursion relation.

Exactly the same approach may be applied to the simple cubic lattice. If the lattice sites are $r=j,k,l$, the red cubes are centered at $r=2(j,k,l)+(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and the blue at $r=2(j,k,l)-(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Here there are 22 basic interactions within the cube instead of the six which occur in the square. Everything else is the same as for $d=2$.

I use the approximate recursion relation to discuss critical behavior by considering the neighborhood of the fixed point. For each p, there is a fixed point $\vec{K}^*(p)$ defined by $\vec{K}^*(p)=R_p{}^L(\vec{K}^*(p))$. To fix p, I use the condition that the best p should make the free energy an extremum for fixed $\vec{K}$. In particular, I write the recursion relations

$$\vec{K}^\alpha=[R_{p+\Delta p}{}^L]^\alpha(K^*(p)).$$

In general $K_0{}^\alpha-K_0(p)$ will grow as $2^{d\alpha}g(p)\Delta p$ for large α and small Δp. (See Wegner's[1] eigenvalue analysis of deviations from criticality.) The extremum condition for the approximation is that $g(p)=0$. In two dimensions this is satisfied at $p=p^*=0.76$.

Now fix p at p^*. Calculate the remaining eigenvalues of the approximate recursion relation. Use the standard[1,2] formulas to calculate the critical indices δ and α from these eigenvalues. The resulting calculated indices are shown in Table I. In the three-dimensional case, the errors in my estimated indices are totally unknown. In two dimensions, I have achieved comparable accuracy with previous work[9] by using a recursion calculation which is very much simpler than that used heretofore.

The procedure outlined in this paper gives better numerical results than one might have expected *a priori*. Why? I am not sure. To investigate this point further, I have applied the varia-

TABLE I. Calculated values of critical indices. The exact values come from the Onsager solution of the two-dimensional Ising model.

	$d=2$		$d=3$	
Parameter	Calculated value	Exact value	Calculated value	"Accepted" value
δ	15.04	15.0	4.818	5.0 ± 0.2
$2-\alpha$	1.998	2.0	1.887	1.875

tional technique of this paper to the Gaussian model, using the same recursion method as used by Bell and Wilson.[10] For the Gaussian model, the present method gives a $\vec{K}^*$ which agrees exactly with the critical coupling strengths and gives precisely the right critical indices for all values of dimensionality. Further study will be required to establish why the errors of this procedure are so unexpectedly small.

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[4]The symmetries actually considered in this paper are translational invariance plus the point group of the square or cube.

[5]R. P. Feynman, *Statistical Mechanics: A Set of Lectures. Richard P. Feynman,* edited by D. Pines (Benjamin, Reading, Mass., 1972), pp. 66–71.

[6]This equation and others assume that all the variables commute. A generalization to noncommuting cases is easily obtained (see Ref. 5).

[7]Proof that $\mathcal{H}^L(\mu) \leq \mathcal{H}'(\mu)$: Let $\mathcal{H}^L(\mu)$ obey the lattice symmetries and be defined by Eq. (7), in which $V(\mu, \sigma)$ is odd under these symmetries. Then the standard methods (Ref. 5) suffice to prove that

$$\mathrm{Tr}_\mu X(\mu) e^{-\mathcal{H}^L(\mu)} \geq \mathrm{Tr}_\mu X(\mu) e^{-\mathcal{H}'(\mu)}$$

for any positive semidefinite $X(\mu)$ which obeys the lattice symmetries. Choose $X(\mu)$ to be projection operator onto all configurations of the μ lattice which are equivalent to μ under the symmetry operations. The desired result then follows.

[8]Such motion was actually used by Kadanoff and Houghton (Ref. 3) to get rid of some "hard" terms.

[9]K. Wilson, private communication; Th. Neimeijer and J. M. J. van Leeuwen, Phys. Rev. Lett. 31, 1412 (1973), and Physica (Utrecht) 71, 1974; M. Nauenberg and B. Nienhuis, Ref. 1; Kadanoff and Houghton, Ref. 3.

[10]T. L. Bell and K. G. Wilson, to be published.

Stability of the Critical Surface in Irradiated Plasma*†

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The linear and nonlinear evolution of an instability of one-dimensional filaments in plasma is investigated. The relevance of these results to two-dimensional breakup of the critical surface in laser-plasma interactions is pointed out.

The possibility of obtaining self-trapped solutions for radiation intensity in media which have a nonlinear index of refraction is well established.[1,2] The question of the stability of these states bears directly on their possible experimental observation. For the case of filamentary equilibria in plasma, it has been shown theoretically that such states are unstable to kink, or bending, perturbations.[3]

We derive here the properties of another, faster growing, instability of such filamentary structures in plasma. This instability is of the sausage, or necking, type. We also demonstrate by numerical solution of the nonlinear coupled electromagnetic and plasma-fluid equations that this instability may result in the destruction of planar symmetry of the critical surface in an expanding irradiated plasma. Such a two-dimensional breakup of the critical surface can modify the collective absorption mechanisms which occur there and which play a central role in many laser-fusion schemes. We note that the existence of this instability in cubically nonlinear media has been demonstrated by Zakharov[4] and Zakharov and Rubenchik.[5] These calculations do not allow a determination of the growth rate, however, because either the inertia of the nonlinearity[4,5] or the parallel dispersion of the trapped wave[5] is neglected. Both effects are important in the case of electromagnetic radiation trapped in plasma.