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Direction of Paired Spins in the A_1 Phase of ^{3}He: A Test of the Paramagnon-Induced–Pairing Hypothesis*

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It is shown within the context of simple paramagnon theory that if spin fluctuations are responsible for the pairing in superfluid ^{3}He, then the direction of paired spins in the A_1 phase should be opposite to that of the field-induced magnetization in the normal state, in contradiction to predictions made using weak-coupling theory. Measurements of the direction of paired spins in the A_1 phase can thus help to test the paramagnon-pairing hypothesis.

The interaction via spin fluctuations of two ^{3}He atoms has been shown to be attractive when the pair is in a state having odd angular momentum.[1] It has been suggested that this attractive interaction is responsible for the two superfluid transitions which have been observed in ^{3}He. Experimental evidence in support of this hypothesis is indirect. Anderson and Brinkman,[2] as well as other groups,[3] have pointed out that spin-fluctuation–induced pairing must be treated within the framework of a new kind of a "strong-coupling" theory: The paramagnons which mediate the pairing must also be modified as a result of the pairing. This "feedback effect" gives rise to a difference in the free energies of the Anderson-Morel and Balian-Werthamer states.[4]

We believe that this recently proposed mechanism for superfluidity is sufficiently novel that it deserves further study and experimental confirmation. It is the purpose of the present note to demonstrate that there exists a concrete experiment which can provide a strong test of the paramagnon-induced–pairing hypothesis. We confine ourselves here entirely to spin-fluctuation theories of the "paramagnon," as distinguished from "Fermi liquid," type in which the enhancement factor is roughly 20 at a pressure of 27 atm. The paramagnon theory of normal ^{3}He is strongly supported by experiments of Meyer and co-workers on the normal-state ^{3}He susceptibility which give an excellent fit to the calculations of Béal-Monod, Ma, and Fredkin[5] *with no adjustable parameters.*

In order to find an experimental test of the paramagnon-induced–pairing hypothesis we derive the gap equations for superfluid ^{3}He in the presence of a uniform, static magnetic field, H, using the spin-fluctuation theory, previously discussed by Anderson and Brinkman,[2] extended to the case $H \neq 0$. From the gap equations the transition temperatures are calculated and the nature of the paired states in the presence of a field is discussed. The paramagnon-induced attractive interaction is field dependent. This field dependence is analogous to the feedback effect mentioned above and results from the fact that the paramagnons, which mediate the pairing, are modified in the presence of a magnetic field. This behavior is in contrast to that which one obtains for ordinary (phonon-induced) superconductivity. Because of these effects, studies of the behavior of the system when $H \neq 0$ should help to provide insight into the pairing mechanism in ^{3}He. In particular, it will be shown below that the predicted direction of alignment of the paired spins in the A_1 phase[6] is different in spin-fluctuation and weak-coupling theories.[7] Thus a measurement of this spin direction should provide a strong test of the spin-fluctuation hypothesis.

The gap equations for paramagnon-induced pairing are obtained by analogy with ordinary phonon-induced superconductivity. The generalized Green's function is related to a generalized self-energy using the equations of motion for the ^{3}He atoms. The self-energy is then written approximately in terms of the Green's function using only the paramagnon diagrams. For the phonon case this scheme is equivalent to the self-consistent Hartree-Fock approximation and because of Migdal's theorem, it is accurate to lowest order in $(m/M)^{1/2}$, where m and M are the electron and ion mass, respectively. For the paramagnon case an additional approximation will be used here for simplicity: the static approximation in which the self-energy is assumed to be frequency independent. Thus the quasiparticle renormalization factor Z is taken to be unity. This follows the approach of Anderson and Brinkman.[2] The validity of this approximation scheme for the case of paramagnon-induced pairing will be discussed in more detail in a later publication.[8]

The generalized Green's functions, defined as

$$\tilde{G}_{\sigma\sigma'}{}^{\rho\rho'}(r,t)=-i\rho'\langle T\psi_\sigma{}^\rho(r,t)\psi_{\sigma'}{}^{\rho'}(r',t')\rangle, \qquad (1)$$

where $\psi_\sigma{}^{+1}=\psi_\sigma$ and $\psi_\sigma{}^{-1}=\psi_\sigma{}^\dagger$, are related to the self-energy $\Sigma_{\sigma\sigma'}{}^{\rho\rho'}(\vec{k})$ as

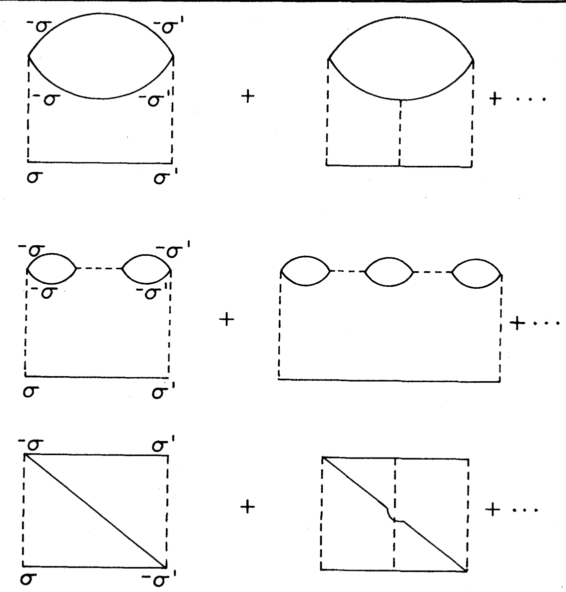

FIG. 1. Diagrams which contribute to the (normal and anomalous) self-energy $\Sigma_{\sigma\sigma'}{}^{\rho\rho'}$ within the paramagnon approximation.

$$\tilde{G}_{\sigma\sigma}{}^{+-}(p)=[\epsilon_k+\omega-\mu_0\sigma H-\Sigma_{\sigma\sigma}{}^{+-}(-k)][\epsilon_k+\omega+\mu_0\sigma H-\Sigma_{-\sigma-\sigma}{}^{+-}(-k)]/D_\sigma(p), \qquad (2)$$

$$\tilde{G}_{\sigma\sigma}{}^{++}(p)=\Sigma_{\sigma\sigma}{}^{++}(\vec{k})[\epsilon_k-\omega+\mu_0\sigma H-\Sigma_{-\sigma-\sigma}{}^{+-}(k)]/D_\sigma(-p), \qquad (3)$$

and

$$\tilde{G}_{-\sigma\sigma}{}^{++}(p)=\Sigma_{-\sigma\sigma}{}^{++}(\vec{k})[\epsilon_k-\omega-\mu_0\sigma H-\Sigma_{\sigma\sigma}{}^{+-}(k)]/D_\sigma(-p), \qquad (4)$$

where

$$D_\sigma(p)=\{[\epsilon_k-w-\mu_0\sigma H-\Sigma_{\sigma\sigma}{}^{+-}(k)][\epsilon_k+\omega-\mu_0\sigma H-\Sigma_{\sigma\sigma}{}^{+-}(-k)][\epsilon_k+\omega+\mu_0\sigma H-\Sigma_{-\sigma-\sigma}{}^{+-}(-k)]$$
$$+|\Sigma_{\sigma\sigma}{}^{++}(\vec{k})|^2[\epsilon_k+\omega+\mu_0\sigma H-\Sigma_{-\sigma-\sigma}{}^{+-}(-k)]+|\Sigma_{\sigma-\sigma}{}^{++}(\vec{k})|^2[\epsilon_k+\omega-\mu_0\sigma H-\Sigma_{\sigma\sigma}{}^{+-}(-k)]\}. \qquad (5)$$

In Eqs. (1)–(5), $\sigma=+1$ (-1) corresponds to spins with z components parallel (antiparallel) to the external field and ϵ_k is the single-particle energy measured relative to the Fermi energy.

In paramagnon theories of ^{3}He, it is assumed that the Hamiltonian of the system consists of a noninteracting term and a phenomenologically derived local interaction which describes the repulsion (of strength I) between opposite-spin atoms. This term leads to an indirect attractive interaction between atoms in the spin-triplet state. To obtain order-of-magnitude agreement between theory and experiment in estimates of the superfluid transition temperature, it *must* be assumed[2] that there is an additional repulsive term, V_0, acting between parallel-spin atoms. The diagrams which are included in obtaining the self-energy within the paramagnon approximation are shown in Fig. 1. It should be noted that the propagators in the diagrams (represented by solid lines) are assumed to depend on the self-energy and on H. This dressing of the Green's functions represents the feedback effect mentioned above. The dashed lines in Fig. 1 represent the interaction I. From these diagrams and Eqs. (2)–(5) we have obtained a set of coupled equations for the components of the self-energy $\Sigma_{\sigma\sigma}{}^{++}$, $\Sigma_{\sigma\sigma}{}^{+-}$, $\Sigma_{-\sigma\sigma}{}^{++}$. In the case of zero magnetic field the ladder sums indicated in Fig. 1 may be summed explicitly to yield equations which are equivalent to those derived independently by Kuroda.[3] In the presence of a magnetic field the equations for the components of Σ are considerably more complicated than in the zero-field case. The discussion of the general field-dependent gap equations will be omitted here. In the remainder of the present paper the field-dependent gap equations in the *small-gap* limit are derived and the A_1 transition temperature is obtained.

In the small-gap limit the normal component of Σ, which appears in the gap equations, is independent of the anomalous component. On performing the frequency sums in the self-energy equations it follows that

$$\Sigma_{\sigma\sigma}{}^{++}(\vec{k}) = \frac{1}{2}\int\frac{d^3k'}{(2\pi)^3}\left[\frac{I^2\chi_1{}^{-\sigma}}{1-I^2\chi_1{}^{\sigma}\chi_1{}^{-\sigma}} - V_0\right]_{k-k'} \tanh\left\{\frac{\beta(\epsilon_{k'} - \Sigma_{\sigma\sigma}{}^{+-} - \sigma\mu_0 H)}{2}\right\}\frac{\Sigma_{\sigma\sigma}{}^{++}(\vec{k}')}{\epsilon_{k'} - \Sigma_{\sigma\sigma}{}^{+-} - \sigma\mu_0 H}, \tag{6}$$

$$\Sigma_{\sigma-\sigma}{}^{++}(\vec{k}) = \frac{1}{2}\int\frac{d^3k'}{(2\pi)^3}\left[\frac{-I}{1-I^2\chi_1{}^{\sigma}\chi_1{}^{-\sigma}} + \frac{I}{1-I\chi_2{}^{\sigma}} - V_0\right]_{k-k'}\Sigma_{\sigma-\sigma}{}^{++}(\vec{k}')$$
$$\times\frac{1}{2}\frac{[\tanh\frac{1}{2}\beta(\epsilon_{k'} - \sigma\mu_0 H - \Sigma_{\sigma\sigma}{}^{+-}) + \tanh\frac{1}{2}\beta(\epsilon_{k'} + \sigma\mu_0 H - \Sigma_{-\sigma-\sigma}{}^{+-})]}{\epsilon_{k'} - \frac{1}{2}(\Sigma_{\sigma\sigma}{}^{+-} + \Sigma_{-\sigma-\sigma}{}^{+-})}, \tag{7}$$

where $\beta = (k_B T)^{-1}$. A similar equation can be written for the normal self-energy $\Sigma_{\sigma\sigma}{}^{+-}(k)$. In the above equations we have neglected the wave-vector dependence of the normal part of the self-energy since the dominant contribution comes from $|k|\sim k_F$, where k_F is the Fermi wave vector, and we have used the fact that

$$|\Sigma_{\sigma-\sigma}{}^{++}(\vec{k})| = |\Sigma_{-\sigma\sigma}{}^{++}(\vec{k})|.$$

The susceptibility functions are given by

$$\chi_1{}^{\sigma}(p) = i\int(2\pi)^{-4}d^4p'\tilde{G}_{\sigma\sigma}{}^{+-}(\vec{k}+\vec{k}', \omega+\omega')\tilde{G}_{\sigma\sigma}{}^{+-}(\vec{k}', \omega'), \tag{8}$$

$$\chi_2{}^{\sigma}(p) = \chi_2{}^{-\sigma}(p) = i\int(2\pi)^{-4}d^4p'\tilde{G}_{\sigma\sigma}{}^{+-}(\vec{k}+\vec{k}', \omega+\omega')\tilde{G}_{-\sigma-\sigma}{}^{+-}(\vec{k}', \omega'). \tag{9}$$

As can be seen from Eqs. (2)–(5), these susceptibilities are clearly H dependent. Considering terms of lowest order in H, it is convenient to define an effective magnetic moment for the system as

$$\mu^{\text{eff}} = \mu_0(1 + d\Sigma_{\uparrow\uparrow}{}^{+-}/d\mu_0 H).$$

Within the RPA[9] $\mu^{\text{eff}} = \mu_0[1 - IN(\epsilon_F)]^{-1}$, where $N(\epsilon_F)$ is the density of states at the Fermi energy. The field-independent part of $\Sigma_{\sigma\sigma}{}^{+-}$ may be included in the Fermi energy and will thus be ignored in the remainder of the paper. It may be seen from Eqs. (8) and (9) that the quantities $(\chi_1{}^{\sigma}\chi_1{}^{-\sigma})$ and $\chi_2{}^{\sigma}$ have no linear terms in H. Hence the leading powers of H which appear in Eqs. (6) and (7) are of order $(\mu_0 H)$ and $(\mu_0 H)^2$, respectively. Since the linearized gap equations represent derivatives of the Landau-Ginzburg free energy, it can be seen that Eqs. (6) and (7) are consistent with the general form of the field-dependent free energy discussed by Ambegaokar and Mermin.[7]

In the Anderson-Morel state it is assumed[4] that the $\vec{k}$ dependence of the anomalous self-energies is given by spherical harmonic functions $Y_{1-1}(\hat{k})$, for $l=1$. By use of the quadratic approximation[2] for the susceptibility $\chi_1{}^{\sigma}$, the $\vec{k}'$ integration in Eq. (6) may be readily performed and the transition temperatures $T_c{}^{\uparrow}(H)$ and $T_c{}^{\downarrow}(H)$ calculated. Following the notation of Ambegaokar and Mermin[7]

$$T_c{}^{\sigma}(H) = 1.14\omega_{\text{sf}}\exp\{-[V_\sigma(H)N(\epsilon_F + \sigma\mu^{\text{eff}}H)]^{-1}\}, \tag{10}$$

where ω_{sf} is the spin-fluctuation frequency and $V_\sigma(H)$ is the effective spin-dependent pairing interaction strength which is obtained from Eq. (6) by decomposing the quantity in the brackets into spherical harmonics and projecting out the $Y_{1-1}(\hat{k})Y_{1-1}{}^*(\hat{k}')$ component:

$$V_\sigma(H) = N(\epsilon_F - \sigma\mu^{\text{eff}}H)I_1 - \sigma\mu^{\text{eff}}(H/\epsilon_F)N(\epsilon_F)I_2 - V. \tag{11}$$

In weak-coupling theories[7] $V_\sigma(H)$ is independent of H and σ. Here V is related to V_0 which is the non-spin-fluctuation contribution to the potential, discussed earlier. For $IN(\epsilon_F)\approx 1$,

$$I_1 = -\tfrac{3}{2}I^2\ln[1 - I^2N^2(\epsilon_F)] \tag{12}$$

and

$$I_2 = \tfrac{1}{2}I^2 + \tfrac{3}{4}I^2[1 - I^2N^2(\epsilon_F)]\ln[1 - I^2N^2(\epsilon_F)]. \tag{13}$$

It can be seen that in the large-enhancement limit $I_1 \gg I_2$. *Thus it follows from Eqs. (10) and (11) that the spin-fluctuation contribution to the field dependence of $T_c{}^{\sigma}(H)$ is essentially zero.*[10] The non-spin-fluctuation part of the interaction which is of the same magnitude as $N(\epsilon_F)I_1$, therefore, determines the field splitting of the A transition. Since the latter interaction is repulsive[2] $(V>0)$, it follows that the direction of spin alignment in the A_1 phase is opposite to that which is predicted using weak-coupling theory. The magnitude of the linear splitting calculated using spin-fluctuation theory with $1-IN(\epsilon_F)=\frac{1}{20}$, $\omega_{sf}=\epsilon_F/20$, and $N(\epsilon_F)V\cong 4.0$ (and the RPA approximation for μ^{eff}) is about 4 times larger than that measured experimentally; in weak-coupling theory, the calculated splitting differs from the experimental one by a factor of more than 25. While there are, at present, no measurements which can verify this prediction, it should shortly be possible[11] to ascertain the direction of the spin pairs in the A_1 state.

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[9]Higher order (in I) terms lead to unphysical expressions for μ^{eff}. It is believed that this behavior arises because of the inapplicability of the static approximation and the neglect of vertex corrections. While more elaborate calculations may be required to incorporate these effects, they are outside the scope of the paramagnon theories which have been previously discussed and which it is intended to explore here.

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Variational Principles and Approximate Renormalization Group Calculations*

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Approximate recursion relations which give upper and lower bounds on the free energy are described. "Optimal" calculations of the free energy can then be obtained by treating parameters within the renormalization equations variationally. As an example, simple lower-bound relations are defined for the two- and three-dimensional Ising models. At the fixed point, a parameter is set variationally, and then critical indices are calculated.

Several recent papers[1] have followed up on the pioneering work of Wilson[2] in using the renormalization group to perform approximate calculations of the free energy. In general, the problem is to calculate a free energy $F_N(\vec{K})$ for a system with N degrees of freedom and a set of coupling constants $\vec{K}$, starting from a Hamiltonian $\mathcal{H}_{\vec{K}}(\sigma)$, in which the σ represents some set of statistical variables.[3] A renormalization transformation $\vec{K}'=R(\vec{K})$ defines a new set of coupling parameters as a function of the old parameters. The $\vec{K}'$ defines couplings in a system with fewer degrees of freedom, $N'<N$, described by a new set of variables which we call μ. The special property of this transformation is that it leaves the total free energy invariant, i.e.,

$$F_N(\vec{K})=F_{N'}(R(\vec{K})). \tag{1}$$

To realize such a transformation, define $F_N(\vec{K}) = -\ln \mathrm{Tr}_\sigma \exp[-\mathcal{H}_{\vec{K}}(\sigma)]$ and construct a new Hamiltonian as

$$\mathcal{H}'(\mu)=-\ln \mathrm{Tr}_\sigma \exp[S(\mu,\sigma)-\mathcal{H}_{\vec{K}}(\sigma)].. \tag{2}$$