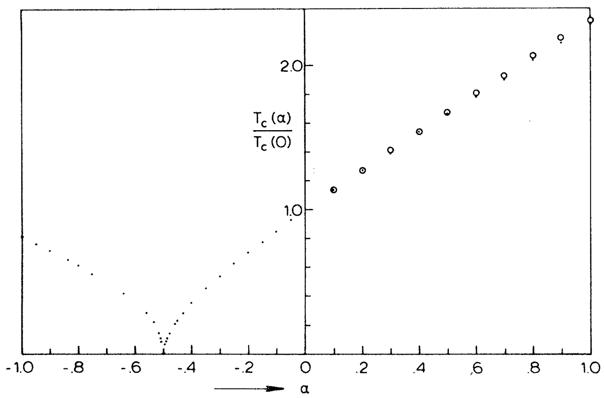

FIG. 3. The critical-temperature ratio $T_c(\alpha)/T_c(0)$ as a function of $\alpha = K_2/K_1$ for $K_1 > 0$, $K_3 = 0$, and $-1 \leq \alpha \leq +1$. Circles correspond to values obtained by Dalton and Wood (Ref. 9) for $0 \leq \alpha \leq +1$.

netic ground-state energy of the Ising spin lattice with only nearest-neighbor and next-nearest-neighbor interactions is the same as the antiferromagnetic ground-state energy with respect to the next-nearest-neighbor sublattices. The cusp can also be understood because these ground states are only degenerate with respect to a change of sign of all the Ising spins. It can be shown that for large values of K_1 we have approximately $K_2 \simeq -\frac{1}{2}K_1 - \frac{1}{2}a\exp(-bK_1)$, which leads to the limit $\ln[(1+2\alpha)/a]T_c(\alpha)/T_c(0) = -bK_1(0)$ as $\alpha \to -\frac{1}{2}$. For $-0.494 \leq \alpha \leq -0.470$ we find an excellent fit to $T_c(\alpha)/T_c(0)$ for $a = 0.1995$ and $b = 0.1241$.

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Spin Waves in Superfluid ^{3}He: Hydrodynamic Regime*

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It is shown that spin waves exist in the hydrodynamic regime in superfluid ^{3}He. Hydrodynamic equations ruling these spin waves are derived. Spin waves are studied in the axial (Anderson-Brinkman-Morel) and in the isotropic (Balian-Werthamer) states.

It has been pointed out recently[1,2] that spin waves must exist in superfluid ^{3}He. The reason is that spin-density fluctuations are coupled to the fluctuations of the direction of the order parameter in spin space; so that spin waves are, in some sense, driven by the fluctuations of the order parameter. This makes it possible for spin waves to exist in superfluid ^{3}He, even without the presence of a magnetic field.

These spin waves have been studied[1,2] in the collisionless regime within the framework of the weak-coupling BCS theory with p-wave pairing. The resulting formulas are complicated, and must, except in some limiting cases, be studied numerically; the formalism is not simple. More important is the fact that the frequency window for this collisionless regime, $1/\tau_D \ll \omega \ll \Delta/\hbar$, is rather small: roughly 1 order of magnitude only, around 100 MHz. This is very inconvenient for experiments.

In this Letter, I want to point out that spin waves also exist in superfluid ^{3}He in the hydrodynamic regime $\omega\tau_D \ll 1$, and that they are described by simple hydrodynamic equations. The frequency window in this regime, $1/T_{1,2} \ll \omega \ll 1/\tau_D$, is extremely large (6 orders of magnitude) and the range of frequency is especially convenient for experiment. This should stimulate experimental work to study these spin waves.

The resulting spin-wave velocity is very simple and can easily be related to the superfluid density[3] [through $\varphi(T)$] and to the susceptibility. Experimental measurements of spin-wave velocities would provide a good independent check for the latter two quantities. Spin waves are also a very good tool to study the microscopic nature of the order parameter, both in spin space and in position space, because the spin polarization of the modes and the anisotropy of their velocity are directly related to the structure of the order parameter. I note that these spin waves are, in some sense, similar to fourth sound: They directly test the superfluid properties because there is no normal part coming in. But they have the main advantage over fourth sound of propagating in the bulk which avoids the problems and uncertainties caused by the restricted geometries required by fourth-sound experiments.

Readers who are not interested in the derivation could skip to the hydrodynamic equations or even directly to the description of the spin-wave modes.

Formalism.—To derive these hydrodynamic equations, I use the method[2] that I have set up to study spin waves in the collisionless regime. The principle is the following: One starts from the kinetic equation for the distribution-function matrix. To take into account the small fluctuations of the direction of the order parameter, one performs space-time-dependent spin rotation, $U = \exp[-i\vec{\sigma}\cdot\vec{\theta}(\vec{r},t)]$, in such a way that, in the new representation, the order parameter is invariant. Then the kinetic equation is expanded in ω and $\vec{q}$, and diagonalized.

Here, to be more general, we will include a static magnetic field $\vec{H}$ in the calculation. This does not at all change the principle of the method. Details on the inclusion of the magnetic field and on the effect of the dipole interaction (which is neglected here) will be given elsewhere. The diagonalized kinetic equation is

$$\omega\,\delta\vec{\nu}_k = \delta\vec{\bar{\nu}}_k q_i \partial E_k/\partial k_i + 2i\,\delta\vec{E}^0 \times \delta\vec{\bar{\nu}}_k, \tag{1}$$

where $\delta\vec{\nu}_k$ is the spin part of the fluctuations of the quasiparticle distribution and $E_k^2 = \xi_k^2 + |\Delta_k|^2$; $\delta\vec{E}^0$ is (in the diagonalized representation) the change in the spin part of the energy matrix due to the magnetic field. Equation (1) reduces in the normal state to the equation used by Silin[4] to study spin waves in normal ^{3}He. $\delta\vec{\bar{\nu}}_k$ is the departure from local equilibrium:

$$\delta\vec{\bar{\nu}}_k = \delta\vec{\nu}_k - \varphi'\,\delta\vec{E}_k, \tag{2}$$

where $\varphi' = \partial\varphi/\partial E_k$ and $2\varphi = -\tanh(\beta E_k/2)$, and $\delta\vec{E}_k$ is the local change in the spin part of the energy matrix due to the change in the quasiparticle distribution and to the fluctuations of the order parameter. We will assume for simplicity that only the s-wave part $F_0{}^a = N_0 f_0{}^a$ of the antisymmetric Fermi-liquid parameters is nonzero. With this simplification[2]

$$\delta\vec{E}_k = k_i\vec{A}_i + (\xi_k/E_k)\vec{X} - (1-\xi_k/E_k)(\vec{d}_k/|\Delta_k|^2)[\vec{d}_k{}^* \cdot (k_i\vec{A}_i - \vec{X})], \tag{3}$$

where $\vec{d}_k$ is related to the order-parameter matrix Δ_k by $\Delta_k = i(\vec{\sigma}\cdot\vec{d}_k)\sigma_y$ and

$$\vec{X} = \vec{V} + f_0{}^a\,\delta\vec{\bar{\rho}}; \tag{4}$$

$\delta\vec{\bar{\rho}}$ is defined below. Here $\vec{A}_i$, which can be thought of as some "spin superfluid velocity," and $-\vec{V}$, which is the "spin part of the chemical-potential shift," are related to the spin rotation by

$$\vec{A}_i = iq_i\vec{\theta}/m^*; \quad \vec{V} = -i\omega\vec{\theta} + \vec{\omega}_0 \times \vec{\theta}; \quad \vec{\omega}_0 = \gamma\vec{H}, \tag{5}$$

where γ is the nuclear magnetic moment of ^{3}He. Equation (5) can be rewritten as

$$m^*\omega\vec{A}_i = im^*\vec{A}_i \times \vec{\omega}_0 - q_i\vec{V}, \tag{6}$$

which is an equation for the spin superfluid velocity. To determine $\vec{A}_i$ and $\vec{V}$, one needs another relation which is the spin-density conservation law:

$$\omega\,\delta\vec{\rho} = q_i\vec{j}_i - i\vec{\omega}_0 \times \delta\vec{\rho}. \tag{7}$$

Here $\delta\vec{\rho}$ is the local change in magnetization and $\vec{j}_i$ the spin current:

$$\delta\vec{\tilde{\rho}} \equiv \delta\vec{\rho} + 2\vec{\theta}\times\vec{\rho}_0 = \sum_k [(\xi_k/E_k)\delta\vec{\nu}_k + (1-\xi_k/E_k)(\vec{d}_k/|\Delta_k|^2)(\vec{d}_k^*\cdot\delta\vec{\nu}_k) + (\varphi/E_k^3)\vec{d}_k\times(\vec{X}\times\vec{d}_k^*)], \tag{8}$$

$$\vec{j}_i = \rho\vec{A}_i + (1/m^*)\sum_k k_i[\delta\vec{\nu}_k - (1-\xi_k/E_k)(\vec{d}_k/|\Delta_k|^2)(\vec{d}_k^*\cdot\delta\vec{\nu}_k) + (\varphi/E_k^3)k_j\vec{d}_k(\vec{A}_i\circ\vec{d}_k)], \tag{9}$$

where ρ is the ^{3}He density; $\delta\vec{\tilde{\rho}}$ is the change in magnetization seen in the rotating spin reference frame. This is not the same as the change in magnetization $\delta\vec{\rho}$ in the rest frame because the static magnetization $\vec{\rho}_0$ gives an oscillating contribution $2\vec{\theta}\times\vec{\rho}_0$ in the oscillating reference frames. Equations (1), (3), (5), and (7)–(9) are our basic equations. They reduce to those given in Ref. 2 for zero magnetic field.

Hydrodynamic equations.—If the collisionless condition $\omega\tau_D \gg 1$ is no longer satisfied, one must add collision terms to Eq. (1). In the hydrodynamic regime $\omega\tau_D \ll 1$, $\delta\vec{\nu}_k$ is given by a local equilibrium condition. Note that there is nothing like "normal spin velocity" because the spin current does not commute with the bare interaction; such a normal velocity would decay by collisions. Therefore the local change in energy is simply $\delta\vec{E}_k$ and the local equilibrium condition is $\delta\vec{\tilde{\nu}}_k = 0$. Equations (8) and (9) become, with the aid of Eqs. (2) and (3),

$$\delta\vec{\tilde{\rho}} = \overleftrightarrow{\chi}\cdot(-\vec{V}), \quad \overleftrightarrow{\chi}^{-1} = \overleftrightarrow{\kappa}^{-1} + f_0^a, \tag{10}$$

$$\vec{j}_i = (\overleftrightarrow{\rho_s})_{ij}\cdot\vec{A}_j, \tag{11}$$

where $\overleftrightarrow{\chi}$ is the static susceptibility tensor at constant $\vec{d}_k$ (with Fermi-liquid effects) and $(\overleftrightarrow{\rho_s})_{ij}$, which is a tensor both in spin and direct space, is a spin superfluid density. Explicitly,

$$\frac{\overleftrightarrow{\kappa}\cdot(-\vec{V})}{N_0} = \int\frac{d\Omega_k}{4\pi}\frac{\vec{d}_k}{|\Delta_k|^2}\times[(-\vec{V})\times\vec{d}_k^*] + \int\frac{d\Omega_k}{4\pi}\int_{-\infty}^{+\infty}d\xi(-\varphi')\frac{\vec{d}_k}{|\Delta_k|^2}[\vec{d}_k^*\cdot(-\vec{V})], \tag{12}$$

$$\vec{j}_i/N_0 = (1/m^*)\int(d\Omega_k/4\pi)\int_{-\infty}^{+\infty}d\xi(k_ik_j/E^2)(\varphi' - \varphi/E)\vec{d}_k\times(\vec{A}_j\times\vec{d}_k^*). \tag{13}$$

Equations (10) and (11) together with Eqs. (7) and (5) [or (6) in zero magnetic field] are the desired hydrodynamic equations. They are very similar to the usual hydrodynamic equations for the particle-density part, especially in zero magnetic field. Equation (10) merely states that, in the local rotating reference frame, the magnetization $\delta\vec{\tilde{\rho}}$ corresponding to the change in spin chemical potential $-\vec{V}$ is given by the susceptibility $\overleftrightarrow{\chi}$. In zero magnetic field or for an isotropic susceptibility, Eq. (10) holds also in the rest frame, that is, between $\delta\vec{\rho}$ and $i\omega\vec{\theta}$. The quantity $i\omega\vec{\theta}$ can be interpreted as the spin chemical-potential change in the rest frame, $-\vec{V}$ being the corresponding quantity in the rotating frame.

Spin-wave modes.—Now we turn to the most important cases, namely the axial (Anderson-Brinkman-Morel) and the isotropic (Balian-Werthamer) states, which are presently believed to describe the A phase and the B phase, respectively. In the axial state, $\vec{d}_k$ has a fixed direction, the y-axis direction for instance. From Eq. (13), we see that a spin superfluid velocity along the y axis gives a zero spin current. Immediately we see that, for zero magnetic field, there is no mode with spin polarization along the y axis. This absence merely reflects the fact that a spin rotation $\vec{\theta}$ along the y axis does not give rise to a fluctuation of the order-parameter direction. This result is also valid in the collisionless regime.[2] There are two degenerate modes with polarization parallel to the x and z axes. Their dispersion relation is easily found to be

$$\omega^2 = \tfrac{1}{3}(qV_F)^2(1+F_0^a)[1-\phi_q(T)], \tag{14}$$

where

$$\phi_q(T) = 3\int(d\Omega_k/4\pi)\int_{-\infty}^{+\infty}d\xi(-\varphi')(\hat{k}\cdot\hat{q})^2. \tag{15}$$

The spin-wave velocity corresponding to Eq. (14) is anisotropic as expected. Note that $\phi_q(T)$ also enters into the expression for the usual superfluid density[3] which can be, in this way, related to the spin-wave velocity.

In nonzero magnetic field, $\vec{d}_k$ is perpendicular to the field. We choose $\vec{\omega}_0$ parallel to the z axis. There are still only two modes but the degeneracy is removed. The longitudinal mode (magnetization along the z axis) is clearly unaffected by the field and Eq. (14) still holds. Because of the Larmor force, the transverse mode is no longer polarized along the x axis. From Eq. (7), $\omega\,\delta\rho_y = i\omega_0\,\delta\rho_x$, and it may also be seen from Eqs. (7) and (10) that $\vec{\theta}$ and $\delta\vec{\tilde{\rho}}$ are parallel. This mode is thus elliptically polarized. The dispersion relation is easily found to be

$$\omega^2 = \omega_0^2 + \tfrac{1}{3}(qV_F)^2(1+F_0^a)[1-\phi_q(T)]. \tag{16}$$

In the isotropic state, $\vec{d}_k = R(\vec{k})$, where R is a fixed rotation. From Eqs. (5), (7), (10), and (11), we obtain

$$\omega^2\vec{\theta} + i\omega\vec{\omega}_0\times\vec{\theta} = \frac{(qV_F)^2}{3}\frac{1-\phi(T)}{\chi(T)}\left[\vec{\theta} - \left\langle 3(\hat{k}\cdot\hat{q})^2\frac{\vec{d}_k(\vec{d}_k\cdot\vec{\theta})}{|\Delta|^2}\right\rangle\right], \tag{17}$$

where $\phi(T)$ is the Yoshida function and $\chi(T)$ is, within some constants, the susceptibility of the isotropic state:

$$\chi(T) = \frac{2+\phi(T)}{3+F_0^a[2+\phi(T)]}. \tag{18}$$

In Eq. (17), the angular average is over the Fermi surface. The eigenmodes of Eq. (17) are in general complicated because there are two symmetry axes in the problem: $\vec{\omega}_0$ and $R(\vec{q})$. However, if $R(\vec{q})$ is parallel to $\vec{\omega}_0$, one finds a longitudinal mode with

$$\omega^2 = \tfrac{2}{15}(qV_F)^2[1-\phi(T)]/\chi(T), \tag{19}$$

and two circularly polarized transverse modes with

$$\omega(\omega \pm \omega_0) = \tfrac{4}{15}(qV_F)^2[1-\phi(T)]/\chi(T). \tag{20}$$

If $R(\vec{q})$ is perpendicular to $\vec{\omega}_0$, one has a mode linearly polarized along the field with

$$\omega^2 = \tfrac{4}{15}(qV_F)^2[1-\phi(T)]/\chi(T), \tag{21}$$

and two transverse modes elliptically polarized. The results in zero field are easily obtained from the first case.

These results agree at zero temperature with the collisionless-regime results,[2] both in the axial and the isotropic state. This is quite natural since, in both regimes, there are no quasiparticle excitations: $\delta\vec{\nu}_k = 0$.

It can easily be seen that these spin waves are not blurred out by spin diffusion. The condition is that, during a period, the spread of the diffusion is much smaller than the wavelength: $q(D/\omega)^{1/2} \ll 1$. With the diffusion constant $D \sim V_F^2\tau_D$ and our results for the spin-wave velocity, we see that this condition is equivalent to the hydrodynamic condition $\omega\tau_D \ll 1$.

The effect of the dipole interaction, in the axial state, is merely to add ω_d^2 to the right-hand side of Eqs. (14) and (16), where ω_d is the NMR shift. In the isotropic state, because of the dipole interaction, the rotation R is around $\vec{\omega}_0$ and one must add $\overleftrightarrow{\Omega}^2\vec{\theta}$ to the right-hand side of Eq. (17); the tensor $\overleftrightarrow{\Omega}$ has a single nonzero eigenvalue which is the longitudinal NMR resonance frequency and corresponds to polarization parallel to the field.

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