Critical Surface for Square Ising Spin Lattice

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The renormalization-group transformations for a square Ising spin lattice are evaluated in a four-cell–cluster approximation to obtain the critical surface in a three-dimensional space of spin-interaction parameters. Agreement is found with known critical curves, particularly Baxter's analytic solution.

The renormalization group[1,2] has been applied with considerable success to evaluate the critical exponents for phase transitions of Ising spin systems. Recently we derived from the renormalization transformations an infinite series expansion for the free energy,[3] and we showed in particular that this series gives an expression for the coefficients of the power-law singularities which had previously not been determined. Extending the cell-cluster approximation of Niemeijer and van Leeuwen,[4] we evaluated as an example the renormalization transformations for a square lattice of sixteen Ising spins, and found that this series leads to thermodynamic functions in good agreement with Onsager's exact solution.[5] In this note we present the results obtained in this approximation for the critical surface in the subspace of nearest-neighbor interactions K_1, next-nearest-neighbor interactions K_2, and four-spin interactions K_3. This surface is of considerable physical interest because it determines the onset of critical transitions including not only ferromagnetic and antiferromagnetic transitions, but also certain ferroelectric transitions, in virtue of the equivalence of a special case of the eight-vertex model with this Ising spin system.[6,7] In particular, we find that the intersection of the critical surface with the plane $K_1=0$ is in good agreement with the Baxter critical curve,[8] while the corresponding intersection with the plane $K_3=0$ agrees with the critical curve obtained by a series-expansion method by Dalton and Wood[9] for $K_1 \geqslant 0$. However, their series expansion fails in the interesting domain $K_1 \geqslant 0$ and $K_2 \leqslant 0$ of competing ferromagnetic and antiferromagnetic couplings, for which new results are given here.

To construct the renormalization transformations, we divide the lattice into Kadanoff cells containing four spins $S_i = \pm 1$, $i=1, 2, 3, 4$, located at the corners of a square. Each cell has then sixteen configurations which can be labeled by the set $(S_1S_2S_3S_4)$. We assign a spin variable $S = \pm 1$ to each cell according to the following rule: The eight configurations $(++++)$ $(-+++)$, $(+-++)$, $(++-+)$, $(+++-)$, $(++--)$, $(+-+-)$, and $(+--+)$ correspond to cell spin up, $S=+1$, while the remaining eight configurations obtain by flipping all the Ising spins correspond to cell spin down, $S=-1$. For the ten configurations with $\sum S_i \neq 0$, this prescription is the same as that given by Niemeijer and van Leeuwen[4] for triangular cells, in conformance with Kadanoff's intuitive ideas. For the six configurations with $\sum S_i = 0$, this choice is one of four possible ways to assign three configurations to $S=+1$ in such a way that the remaining three configurations for $S=-1$ are obtained by flipping the cell Ising spins. These four choices are all equivalent for clusters of a finite size arranged in a square lattice. The smallest cluster satisfying this requirement is a 4×4 square Ising lattice which we have considered in detail.

We have evaluated numerically the renormalization transformations $K_\alpha' = F_\alpha(K_1, K_2, K_3)$, $\alpha = 1, 2, 3$, and have found a single unstable fixed point at $K_1^* = 0.300$, $K_2^* = 0.0871$, and $K_3^* = -0.001\,26$ which determines the critical behavior, and two stable fixed points at $K_\alpha = 0$ and at $K_\alpha \to \infty$, which determine the asymptotic end point of transformations above and below the critical surface. This surface, which is defined by the subset of points which are mapped into the unstable fixed point, is shown in Fig. 1 for the domain $-2 \leqslant K_1, K_3 \leqslant +2$ and is seen along the direction $K_1=1$, $K_2=1$, and $K_3=-1$. It consists of two branches which are joined in the limit $K_1 \to \pm\infty$, $K_2 = -\frac{1}{2}|K_1|$, with K_3 fixed. Special points $\overline{K}_\alpha \neq K_\alpha^*$ on this surface map directly into the unstable fixed point, i.e., $\overline{K}_\alpha = F_\alpha(K^*)$, and therefore we call them *fixed-point images* because these points play an analogous role to fixed points in describing the critical transitions nearby. The symmetry of the critical surface and the existence of these fixed-point images can be understood from the sym-

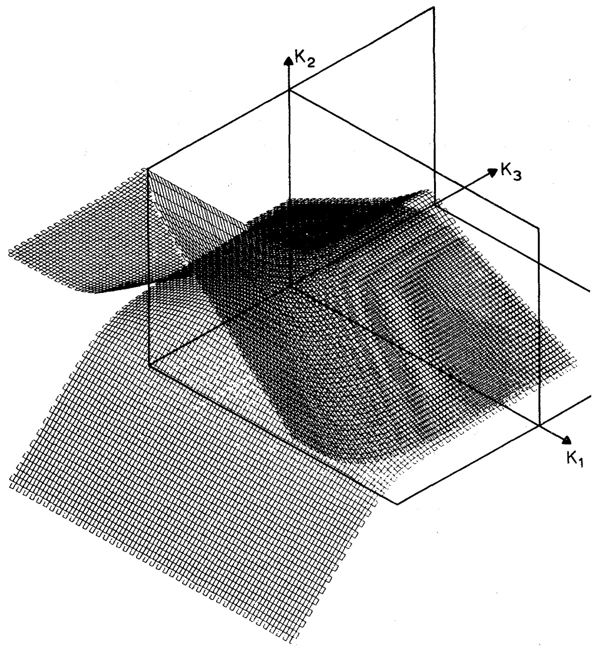

FIG. 1. Critical surface in the range $-2 \leq K_1, K_3 \leq +2$, seen along the direction $K_1=1$, $K_2=1$, and $K_3=-1$.

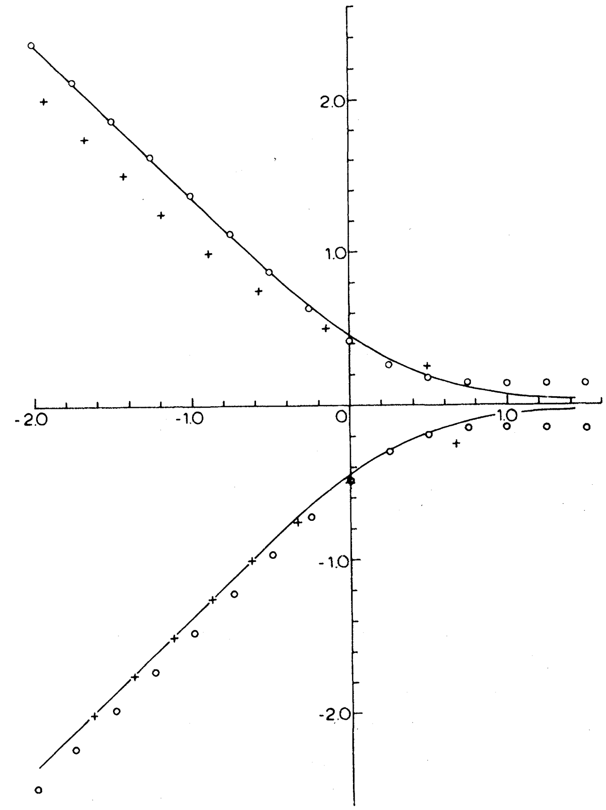

FIG. 2. Intersection of the upper and lower branches of the critical surface with the plane $K_1=0$, shown by circles, and intersection of the upper branch with the plane $K_2=0$, shown by crosses, as a function of K_3. The solid curve is Baxter's critical curve $\exp(2K_3) = |\sinh 2K_2|^{-1}$.

metry of the free energy under the transformation $K_1 \to -K_1$, $K_2 \to K_2$, $K_3 \to K_3$, and the transformation $K_2 \to -K_2$, $K_3 \to K_3$, with $K_1=0$. Actually, these symmetries are only approximately satisfied by our renormalization transformations for a four-cell cluster. It is possible to implement these symmetries exactly by other choices of cell spin assignments. However, we have found in these cases at least one cell configuration with $\sum S_i \neq 0$ which has the sign of $\sum S_i$ opposite to that of the cell spin, and the agreement of the corresponding critical values with known results is not very good.

The intersections of the upper and lower branches of the critical surface with the plane $K_1=0$ are shown by circles in Fig. 2 and compared with the critical curve $\exp(2K_3) = |\sinh 2K_2|^{-1}$ obtained by Baxter for the eight-vertex model.[8] The crosses in Fig. 2 show the intersection of the upper branch of the critical surface with the plane $K_2=0$, which seems not to have been calculated before. It appears that Baxter's critical curve may also be applicable in this case. It has been pointed out[7] that Baxter's analytic solution implies a continuous line of fixed points depending on the value of K_3, while in our approximation we found a unique unstable fixed point. If we consider the subset of transformations $K_\alpha' = F_\alpha(K)$ for $\alpha = 1, 2$, keeping K_3 fixed, we found the dependence of the critical exponent α on K_3 for $-\frac{1}{2} \leq K_3 \leq 0$ in reasonable agreement with Baxter's result, but not for other values of K_3. Although this approach may not be self-consistent, it does suggest that a larger size cluster may lead to better results.

Finally, we have evaluated the dependence of the critical temperature $T_c(\alpha)$ on the ratio $\alpha = K_2/K_1$ for $K_3=0$ and $K_1>0$ from the intersection of the critical curve with the plane $K_3=0$. The results are shown in Fig. 3 for $-1 \leq \alpha \leq 1$, where we include for comparison the values obtained by Dalton and Wood for $0 \leq \alpha \leq 1$.[9] In the interesting domain $\alpha \leq 0$, where the series-expansion methods fail, we find that $T_c(\alpha)$ has a cusp and vanishes at $\alpha = -\frac{1}{2}$. At this value of α the ferromag-

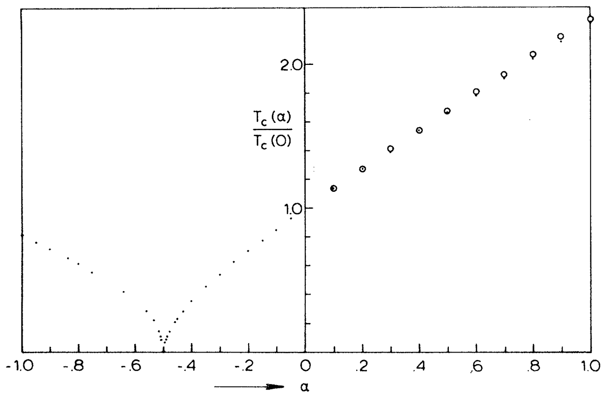

FIG. 3. The critical-temperature ratio $T_c(\alpha)/T_c(0)$ as a function of $\alpha = K_2/K_1$ for $K_1 > 0$, $K_3 = 0$, and $-1 \leq \alpha \leq +1$. Circles correspond to values obtained by Dalton and Wood (Ref. 9) for $0 \leq \alpha \leq +1$.

netic ground-state energy of the Ising spin lattice with only nearest-neighbor and next-nearest-neighbor interactions is the same as the antiferromagnetic ground-state energy with respect to the next-nearest-neighbor sublattices. The cusp can also be understood because these ground states are only degenerate with respect to a change of sign of all the Ising spins. It can be shown that for large values of K_1 we have approximately $K_2 \simeq -\frac{1}{2}K_1 - \frac{1}{2}a\exp(-bK_1)$, which leads to the limit $\ln[(1+2\alpha)/a]T_c(\alpha)/T_c(0) = -bK_1(0)$ as $\alpha \to -\frac{1}{2}$. For $-0.494 \leq \alpha \leq -0.470$ we find an excellent fit to $T_c(\alpha)/T_c(0)$ for $a = 0.1995$ and $b = 0.1241$.

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Spin Waves in Superfluid ^{3}He: Hydrodynamic Regime*

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It is shown that spin waves exist in the hydrodynamic regime in superfluid ^{3}He. Hydrodynamic equations ruling these spin waves are derived. Spin waves are studied in the axial (Anderson-Brinkman-Morel) and in the isotropic (Balian-Werthamer) states.

It has been pointed out recently[1,2] that spin waves must exist in superfluid ^{3}He. The reason is that spin-density fluctuations are coupled to the fluctuations of the direction of the order parameter in spin space; so that spin waves are, in some sense, driven by the fluctuations of the order parameter. This makes it possible for spin waves to exist in superfluid ^{3}He, even without the presence of a magnetic field.

These spin waves have been studied[1,2] in the collisionless regime within the framework of the weak-coupling BCS theory with p-wave pairing. The resulting formulas are complicated, and must, except in some limiting cases, be studied numerically; the formalism is not simple. More important is the fact that the frequency window for this collisionless regime, $1/\tau_D \ll \omega \ll \Delta/\hbar$, is rather small: roughly 1 order of magnitude only, around 100 MHz. This is very inconvenient for experiments.