Phys. B68, 585 (1974).
[10]G. Soff, B. Müller, and J. Rafelski, University of Pennsylvania Report No. UPR-0033N, 1974 (to be published).
[11]These points have also been discussed by Rafelski, Müller, and Greiner (Ref. 9) using different methods.

General Derivation of Bäcklund Transformations from Inverse Scattering Problems*

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A general way is demonstrated to derive Bäcklund transformations for nonlinear partial differential equations that are solvable by the inverse scattering method in the scheme of Ablowitz, Kaup, Newell, and Segur.

Recently, Ablowitz *et al.*[1] discovered a general scheme for finding the set of nonlinear partial differential equations that are solvable by the inverse scattering method. This paper will show how one can derive the Bäcklund transformation from the auxiliary equations for the inverse problem.[2] This derivation provides the basis for unifying the two different approaches to solving these nonlinear equations.

Ablowitz *et al.*[1] have found that the integrability conditions for the systems of linear partial differential equations

$$v_{1x}+i\zeta v_1=qv_2,\quad v_{2x}-i\zeta v_2=rv_1 \tag{1}$$

and

$$v_{1t}=Av_1+Bv_2,\quad v_{2t}=Cv_1-Av_2 \tag{2}$$

are exactly those equations which allow soliton solutions solvable by the inverse scattering method. The integrability conditions are

$$\begin{aligned}&A_x=qC-rB,\quad B_x+2i\zeta B=q_t-2Aq,\\&C_x-2i\zeta C=r_t+2Ar.\end{aligned} \tag{3}$$

Finite expansions of A, B, and C in terms of ζ reduce the problem to specific equations of interest, for example, Korteweg-de Vries (KdV),[2] modified Korteweg-de Vries (mKdV),[3] sine-Gordon,[4] and nonlinear Schrödinger equations.[5]

Equations that are solvable by an inverse scattering problem are found to be also solvable by Bäcklund transformations.[6,7] But till now only two such transformations have been found.[6,7] They are derived independently from a tedious *ad hoc* elimination procedure (or simply from a guess). I present in this section a unified way of finding them from the inverse problem, and therefore, provide a basis for the statement that corresponding to each inverse problem there exists a Bäcklund transformation.

From Eqs. (1)-(3), we can easily get a system of equations for the quantity $u\equiv v_1/v_2$:

$$u_x=2i\zeta u-ru^2+q,\quad u_t=2Au-Cu^2+B. \tag{4}$$

This equation is very important. I will demonstrate in the following by showing examples that all Bäcklund transformations can indeed be reduced to (or derived from) this set of Riccati equations. A specific identification of u as a functional of q and q' will provide a Bäcklund transformation to a particular differential equation. These Bäcklund transformations can be divided into different classes.

Class I.—For the first class $r=\text{const}=-2$. Equation (4) then becomes, with $i\zeta=k$,

$$u_x=-2ku+2u^2+q,\quad u_t=2Au-Cu^2+B. \tag{5}$$

The simplest example in this class is the Korteweg-de Vries equation,[7] $q_t+12qq_x+q_{xxx}=0$. Following Ablowitz *et al.*, we identify A, B, and C to be

$$\begin{aligned}&A=4k^3+4kq-2q_x,\\&B=-4k^2q+2kq_x-q_{xx}-4q^2,\\&C=8k^2=8q.\end{aligned} \tag{6}$$

If we eliminate q in Eq. (5) we get an equation for u,

$$u_t-24u^2u_x+24kuu_x+u_{xxx}=0. \tag{7}$$

This is a mixed KdV-mKdV equation. Equation (5) provides a Bäcklund transformation between solutions of the KdV equation and Eq. (7). In particular, if $k=0$, Eq. (7) reduces to the pure mKdV equation and Eq. (5) becomes the famous Miura transformation. Now, we can see that if (u,k) satisfies Eq. (7), then $(-u,-k)$ also satisfies

Eq. (7). This gaugelike invariance of Eq. (7) tells us immediately that a second solution q' exists for the KdV equation, such that

$$-u_x=-2ku+2u^2+q' \tag{8a}$$

$$-u_t=-2A(q',-k)u-C(q',-k)u^2+B(q',-k) \tag{8b}$$

From Eqs. (5) and (8), we get

$$2u_x=q-q', \tag{9a}$$

$$q+q'=4ku-4u^2 \tag{9b}$$

and

$$2u_t=2u(A+A')-u^2(C-C')+(B-B'), \tag{10a}$$

$$0=2u(A-A')-u^2(C+C')+(B+B'). \tag{10b}$$

Two different forms of Bäcklund transformations can be derived from Eq. (9).

(i) Let $q=w_x$; then $u=\frac{1}{2}(w-w'+k)$, and we get the Bäcklund transformation between w and w':

$$(w+w')_x=k^2-(w'-w)^2, \tag{11a}$$

$$(w-w')_t=2A(w-w'+k)-\tfrac{1}{2}C(w-w'+k)^2+2B, \tag{11b}$$

where w satisfies the equation

$$w_t+6w_x{}^2+w_{xxx}=0.$$

(ii) Solving for u from Eq. (9b), we get

$$u=\tfrac{1}{4}\{2k\pm[4k^2-4(q+q')]^{1/2}\}.$$

Because Eq. (9b) is a local equation, we can choose a constant x_0 such that q would be a continuous soliton solution and then

$$u=\tfrac{1}{2}\{k\pm[k^2-(q+q')]^{1/2}H(x-x_0-4k^2t)\}, \tag{12}$$

where H is the Heaviside step function. Substituting this into Eq. (9a) we get the Bäcklund transformation between q and q' directly,

$$(q+q')_x=\mp 2(q-q')[k^2-(q+q')]^{1/2}H(x-x_0-4k^2t), \quad u_t=2Au-Cu^2+B. \tag{13}$$

I make two remarks here. First, Eq. (10b) is equivalent to Eq. (9a) plus (9b), so that either Eq. (10a) or (8b) with the appropriate u can serve to be the time part of the Bäcklund transformation. Second, the two forms of Bäcklund transformations, Eqs. (11) and (13), are actually equivalent. I do not wish the reader to get the wrong impression that the Bäcklund transformation is not unique.

Class II.—$r=-q$, $i\zeta=k/2$. In this class, we can also get two equivalent forms of Bäcklund transformations. Let us consider two examples.

(a) For the mKdV equation, $q_t+6q^2q_x+q_{xxx}=0$,

$$A=\tfrac{1}{2}k^3+kq^2, \quad B=-k^2q+kq_x-q_{xx}-2q^3, \quad C=k^2q+kq_x+q_{xx}+2q^3. \tag{14}$$

Now, eliminating q, we get the following equation for u:

$$u=\tan\tfrac{1}{2}v, \quad v_t+v_{xxx}+\tfrac{1}{2}v_x{}^3+\tfrac{1}{2}3k^2v_x\sin^2 v. \tag{15}$$

This equation also possesses the invariance $(u,k)\rightarrow(\pm u,\pm k)$. It results in the following self-Bäcklund transformations for the mKdV equation: (i) $v=w\mp w'$,

$$(w\pm w')_x=k\sin(w\mp w'), \tag{16a}$$

$$(w\pm w')_t=-2w_x{}^2k\sin(w\mp w')\mp 2w_{xx}k\cos(w-w')\mp k^3\sin(w\mp w')\pm 2k^2w_x; \tag{16b}$$

and (ii)

$$u=\{k\pm[k^2-(q\mp q')^2]^{1/2}H(x-x_0-k^2t)\}/(q\mp q'), \tag{17}$$

$$(q\mp q')_x=\mp(q\pm q')[k^2-(q\mp q')^2]^{1/2}H(x-x_0-k^2t),\quad u_t=2Au-Cu^2+B. \tag{18}$$

(b) For the sine-Gordon equation, $2w_{xt}=\sin 2w$,

$$A=-\cos 2w/2k,\quad B=C=q_t/k. \tag{19}$$

The equation satisfied by u is

$$u=\tan\tfrac{1}{2}v,\quad v_{xt}=[1-k^2v_t{}^2]^{1/2}\sin v. \tag{20}$$

It is also invariant under the transform $(u,k)\to(\pm u,\pm k)$. The Bäcklund transformations are then (i) $v=w\mp w'$,

$$(w\pm w')_x=k\sin(w\mp w'),\quad (w\mp w')_t=k^{-1}\sin(w\pm w'); \tag{21}$$

and (ii) the same as Eqs. (17) and (18) but with appropriate A, B, C, and H.

Class III.—$r=-q^*$, $\zeta=\xi+i\eta$. In this class

$$u_x=-2i\zeta u+q^*u^2+q,\quad u_t=2Au-Cu^2+B. \tag{22}$$

The simplest example is the nonlinear Schrödinger equation, $iq_t+q_{xx}+2q^2q^*=0$. For this equation, we have

$$A=-2i\zeta^2+iqq^*,\quad B=+2q\zeta+iq_x,\quad C=-2q^*\zeta+iq_x{}^*. \tag{23}$$

After the elimination of q and q^* in Eq. (22), we get a nonlinear partial differential equation for u and u^*. It is straightforward to show that this equation for u is invariant under the gaugelike transformation

$$(u,\zeta)\to(\pm u,\zeta^*).$$

Therefore, we have

$$\pm u_x=\mp 2i\zeta^*u+q^{*\prime}u^2+q', \tag{24a}$$

$$\pm u_t=\pm 2A(q',\zeta^*)u-C(q',\zeta^*)u^2+B(q',\zeta^*). \tag{24b}$$

From Eqs. (23) and (25) we have

$$0=4\eta u+(q\mp q')^*u^2+(q\mp q'), \tag{25a}$$

$$2u_x=-4i\xi u+(q\pm q')^*u^2+(q\pm q'). \tag{25b}$$

Equation (25b) is not integrable as in the case of the KdV and mKdV equations by the transformation $q=w_x$. We get therefore only one form of Bäcklund transformation for the nonlinear Schrödinger equation itself:

$$u=\{-2\eta\pm[4\eta^2-|q\mp q'|^2]^{1/2}H(x_0-x_0+4\xi t)\}/(q\mp q')^*, \tag{26}$$

$$(q\pm q')_x=-2i\xi(q\pm q')+(q\mp q')[4\eta^2-|q\pm q'|^2]^{1/2}H(x-x_0+4\xi t), \tag{27a}$$

$$(q\pm q')_t=-2i\xi(q\pm q')_x+i(q\mp q')_x[4\eta^2-|q\pm q'|^2]^{1/2}H(x-x_0+4\xi t)+i(q\pm q')(qq^*+q'q'^*). \tag{27b}$$

I remark here that class II is only a special subclass of class III. That is, when $\zeta=-ik/2$ is purely imaginary and $q=q^*$ is purely real, the Bäcklund transformations (27a) would reduce to Eq. (18a).

A complex form of the mKdV equation, $q_t+6qq^*q_x+q_{xxx}=0$, exists as a member in class III and shares the same spatial part of the Bäcklund transformation as the nonlinear Schrödinger equation; so does the Hirota equation[8] $q_t+6\alpha qq^*q+\alpha q_{xxx}+i\beta q_{xx}+2i\beta q^2q^*=0$.

It is demonstrated above that from the inverse scattering problem defined by Eqs. (1)–(3), Bäcklund transformations between different classes of nonlinear partial differential equations can be found. The existence of gaugelike invariance for the equation satisfied by (u, ξ) makes it possible to find a self-Bäcklund transformation for q and q'. These Bäcklund transformations can be divided into classes. Equations in the same class have an identical spatial part of their Bäcklund transformations. Their solutions therefore satisfy the same superposition formula. For example the mKdV and sine-Gordon equations have the same superposition formula,

$$\tan\frac{w_3-w_0}{2}=\frac{k_1+k_2}{k_1-k_2}\tan\frac{w_1-w_2}{2}.$$

This formula renders it possible to construct N-soliton solutions by algebraic manipulations only.

Further generalizations to higher-order inverse problems of what has been done above is possible. I will report some examples in another paper.

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[1]M. J. Ablowitz, P. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 31, 125 (1973).

[2]C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1969).

[3]M. Wadati, J. Phys. Soc. Jpn. 34, 1289 (1973).

[4]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973).

[5]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. JETP 34, 62 (1972)].

[6]G. L. Lamb, Jr., Rev. Mod. Phys. 43, 99 (1971).

[7]H. D. Wahlquist and F. B. Estabrook, Phys. Rev. Lett. 31, 1386 (1973).

[8]R. Hirota, J. Math. Phys. (N.Y.) 14, 805 (1973).

Total Cross Sections of p and $\bar{p}$ on Protons and Deuterons between 50 and 200 GeV/c*

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Proton and antiproton total cross sections on protons and deuterons have been measured at 50, 100, 150, and 200 GeV/c. The proton cross sections rise with increasing momentum. Antiproton cross sections fall with increasing momentum, but the rate of fall decreases between 50 and 150 GeV/c, and from 150 to 200 GeV/c there is little change in cross section.

We have measured p and $\bar{p}$ total cross sections on protons and deuterons in 50-GeV/c steps between 50 and 200 GeV/c. The experiment, which was carried out in the $M1$ beam[1,2] at the Fermi National Accelerator Laboratory, used a "good geometry" transmission technique.

Incident particles were defined by scintillation counters and identified by two differential gas Cherenkov counters,[3] allowing cross sections of two different particles to be measured simultaneously; in addition, a threshold gas Cherenkov counter[4] could be used in anticoincidence when required. Contamination of unwanted particles in the selected p and $\bar{p}$ beams was always below 0.1%.

The 3-m-long liquid hydrogen and deuterium targets and an identical evacuated target were surrounded by a common outer jacket of liquid hydrogen for temperature stability.[5] By continuously monitoring the vapor pressure in the outer jacket, the target temperature and therefore the hydrogen and deuterium densities were determined[6]; density variations were less than 0.07% throughout the experiment. Target lengths were