where L is the inductance of the circuit. If L can be made as small as 10^{-7} H, this time constant $\simeq 10^{-4.5}$ sec; so the detection system should be designed to measure the direction of the initial current in that time interval.

Clearly the experiment is a difficult one, involving the making and breaking of contacts that serve as weak couples.[6] But the difficulties do not seem insurmountable. It should be worth doing not only from the viewpoint of superselection rules, but also from the viewpoint of getting a clearer understanding of the reality of the phase of a disjoint superconductor, not just relative phases between coupled local regions.

If "phase retention" by a superconductor in space-time isolation from the sink can be experimentally established, not only the charge superselection rule loses fundamental significance, but the lepton number superselection rule does also. It is then difficult to conceive that the baryon number superselection rule is more fundamental (even though experimentally it would be harder to test the proposition). One may ask in that case whether there is any superselection rule in nature at all. In this connection we wish to note that the univalence superselection rule seems to be in a different category. Although in Ref. 1 *gedanken* experiments were discussed as to how the change in phase of a fermion under a 360° rotation may be "measured," it seems to us that as long as the connection between spin and statistics remains valid, operators connecting different univalent sectors cannot be local observables without violating causality. Since the electron field ψ smeared with test functions localized in spacelike separated regions do not commute, ψ cannot be an observable, whereas $\psi\psi$ can be if the experiment proposed here has a positive result.

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[5]In fact, for large separation $C \simeq C_1C_2/(C_1+C_2)$, where C_1 and C_2 are the self-capacitances. Therefore the two superconductors may be arbitrarily far apart and still retain the relative phase if their self-capacitances are both large enough.

[6]Point contact junctions have been extensively studied since they were first applied to Josephson effect experiments [J. E. Zimmerman and A. H. Silver, Phys. Lett. 10, 47 (1964)]; we thank T. Bedard and N. Walker for a discussion of some experimental aspects of such junctions. Other ways of switching should also be explored; the arrangement proposed here is illustrative and may not be the most effective one.

Vacuum Polarization in Heavy-Ion Collisions*

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The results of a study on vacuum polarization, orders $\alpha(Z\alpha)^n$, $n \geq 3$, for large-Z systems encountered in heavy-ion collisions are presented. It is shown that the higher-order vacuum polarization cannot prevent the $1S_{1/2}$ state from reaching the lower continuum, $E = -m_ec^2$, for some critical charge $Z_{cr} \sim 170$. In addition, the stability and localization of a heliumlike system for $Z > Z_{cr}$ is demonstrated.

An interesting application of heavy-ion collisions is to the study of quantum electrodynamics of strong fields. For short times, at least, systems with large effective charge Z will be formed with $Z\alpha > 1$. In the strong fields of such systems, highly relativistic electronic bound states are expected to occur with binding energies B exceeding the electron rest mass m_e, and for some critical charge, $Z_{cr} \sim 170$, the $1S_{1/2}$ state is expected to reach the lower continuum with $B = 2m_e$.[1] For $Z > Z_{cr}$, it has been predicted[2,3] that spontaneous e^+e^- pair production will occur with the subse-

quent capture of two electrons into a tightly bound heliumlike state and the ejection of two positrons into continuum states. These predictions have been based on solutions of the Dirac equation for finite-size nuclei in which radiative corrections such as vacuum polarization (VP) and the Lamb shift are ignored. For these predictions to be applicable to heavy-ion collisions, it is essential to show that radiative corrections are indeed negligible for large $Z \sim Z_{cr}$ and thus cannot prevent the $1S_{1/2}$ state from reaching the lower continuum. Several qualitative arguments[2,3] and a model calculation based on effective limiting-field Lagrangians[4] suggest that the effect of VP does remain small up to Z_{cr}. Yet a complete quantum electrodynamic calculation had not been carried out to all orders in $Z\alpha$. The purpose of this Letter, then, is to present the results of a numerical calculation of the higher-order VP, orders $\alpha(Z\alpha)^n$, $n \geq 3$, based on the methods of Wichmann and Kroll.[5] The results confirm that VP remains a small perturbation even up to Z_{cr}.

Although the effect of higher-order VP is always much less than that of the first-order (Uehling) potential in atoms with $Z \leq 100$, the results of Wichmann and Kroll[5] for a point nucleus show that the size of the higher-order VP increases sharply near $Z\alpha = 1$. If the VP charge accumulated at the origin for orders $\alpha(Z\alpha)^n$, $n \geq 3$, is denoted by $Q_{WK}{}^{3+}$, then while $Q_{WK}{}^{3+}$ is finite and much smaller than the nuclear charge when $Z\alpha = 1$ ($|Q_{WK}{}^{3+}| \approx 0.05|e| \ll 137|e|$), $dQ_{WK}{}^{3+}/dZ = -\infty$ at $Z\alpha = 1$. Furthermore, the infinite slope of $Q_{WK}{}^{3+}$ can be seen to come from the lowest-angular-momentum [$k = \pm(j + \frac{1}{2}) = \pm 1$] contribution to the VP density. The higher-angular-momentum ($|k| \geq 2$) contribution to $Q_{WK}{}^{3+}$ is seen to vary smoothly past $Z\alpha = 1$ until $Z\alpha = 2$, where the $|k| = 2$ contribution becomes singular. Of course, these singularities in the VP charge density arise because of the assumed point structure of the nucleus. Nevertheless, these results indicate that we may expect a rapid increase in the higher-order VP for $Z\alpha > 1$ when the finite size of the nucleus is taken into account. Furthermore, the fastest growing part should be that due to the lowest-angular-momentum ($|k| = 1$) electron loops. Another important property that can be deduced from the results of Ref. 5 is the relative size of the $|k| = 1$ to the $|k| \geq 2$ contributions to the VP charge density. The $|k| \geq 2$ contribution was found to be always less than 10% of the $|k| = 1$ contribution for orders $\alpha(Z\alpha)^n$, $n \geq 3$, for all $Z\alpha \leq 1$.[6,7] Therefore, a good approximation in the calculation of higher-order VP, relevant for heavy-ion collisions, is obtained by calculating the $|k| = 1$ contribution exactly for finite-size nuclei while neglecting the $|k| \geq 2$ contribution. In this way the dominant and fastest growing part of the VP density is calculated exactly while leaving, in the final results, an overall uncertainty of less than 10% due to neglect of the $|k| \geq 2$ contribution.

The calculation of the VP density, ρ_{VP}, involves an energy contour integral of the trace of the Green's function, TrG, for the Dirac equation.[5] For $Z < Z_{cr}$, the choice of this contour is clear and is given by C_0 in Fig. 1. With this contour, ρ_{VP} is equal to the vacuum expectation value of the Heisenberg current operator, $\langle 0|J_0(x)|0\rangle$. Thus to first order in α and to all orders in $Z\alpha$, ρ_{VP} can be written formally as

$$\rho_{VP} = \tfrac{1}{2}|e|\,(\textstyle\sum_+ \psi_+{}^2 - \sum_- \psi_-{}^2), \qquad (1)$$

where $\psi_\pm$ refer to the positive- and negative-energy eigenfunctions of the Dirac equation.[8] Here positive energy refers to all eigenvalues greater than $-m_e$. In particular, the $1S_{1/2}$ state is contained in the first sum. The heliumlike charge density, ρ_{He}, is defined as the expectation value of the current operator for a state of two electrons in the $1S_{1/2}$ state: $\langle 2e^-(1S)|J_0(x)|2e^-(1S)\rangle$. This density is related to ρ_{VP} by

$$\rho_{He} = -2|e|\psi_{1S_{1/2}}{}^2 + \rho_{VP}, \qquad (2)$$

which is equivalent to the addition of a counterclockwise contour around the $1S$ pole to C_0 in Fig. 1. The sum of these contours may then be deformed to C_{He}. For $Z < Z_{cr}$, then, ρ_{VP} and ρ_{He} are computed via a contour integral along paths C_0 and C_{He}, respectively (in units of $|e|$).

For $Z > Z_{cr}$, the $1S_{1/2}$ pole moves off the physical sheet through the branch point of the lower continuum. Since the vacuum around the bare nucleus can then decay into a heliumlike state plus two free positrons,[2,3] it is natural to rede-

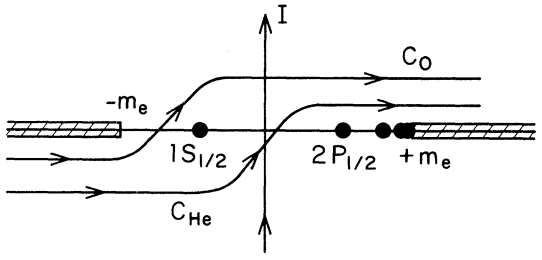

FIG. 1. Singularities of the Green's function in the complex energy plane and contours C_0, C_{He}, and I giving the VP and heliumlike charge densities in units of $|e|$.

fine the vacuum to correspond to the heliumlike state.[9] The VP density, ρ_{VP}, is thus defined to equal ρ_{He} for $Z > Z_{cr}$. While Eq. (2) is no longer meaningful for $Z > Z_{cr}$, the contour-integral representation for ρ_{He} around path C_{He} in Fig. 1 is still well defined. The charge distribution of the overcritical vacuum is thus calculated with contour C_{He}.

The higher-order VP density, ρ^{3+}, for $k = \pm 1$ and $Z < Z_{cr}$ is calculated by expanding TrG in terms of radial Green's functions, $\mathrm{Tr}G_k$, and removing the first-order Green's functions, $\mathrm{Tr}G_k^1$. Isolating the terms corresponding to $k = \pm 1$, the contour integral over path C_0 is performed. In practice, C_0 is deformed to the imaginary axis I, picking up the negative of the residues of all poles of $\mathrm{Tr}G_{k=\pm 1}$ which lie between zero and $-m_e$. The negative of those residues are, of course, the modulus squared of the normalized bound-state wave functions. The calculation of ρ_{He} involves adding to the contour integral along I all but the squares of the $1S_{1/2}$ wave functions with $m_e < B < 2m_e$.

In addition to ρ^{3+}, the third-order, $\alpha(Z\alpha)^3$, VP density, ρ^3, is calculated for $k = \pm 1$ and $Z < Z_{cr}$ to provide a check on internal consistency and to estimate the dependence of ρ^{3+} on different nuclear charge densities. The calculation of ρ^3 involves the contour integral of the third-order Green's function, $\mathrm{Tr}G_k^3$, for $k = \pm 1$ along contour I. Two models for the nuclear charge densities were used for calculating ρ^3: (I) a shell density, $\rho_{Nuc} = \delta(r - R)/4\pi R^2$, and (II) a uniform density, $\rho_{Nuc} = \theta(R - r)/(4\pi R^3/3)$. The nuclear radius was chosen to be 10 fm in both models. The densities ρ^{3+} and ρ_{He} were calculated with model I. The construction of $\mathrm{Tr}G_k$, $\mathrm{Tr}G_k^1$, and $\mathrm{Tr}G_k^3$ is the same here as in Ref. 6 and will be discussed in more detail in a subsequent paper.

The following tests check the numerical accuracy of the constructed Green's functions. First, the location of the $1S_{1/2}$ and $2P_{1/2}$ poles were computed as a function of Z. The values of $(Z\alpha)_{cr}$ were determined for model-I nuclei for $R = 8$, 10, and 12 fm, with $(Z\alpha)_{cr} = 1.251\,89$, $1.274\,59$, and $1.295\,30$, respectively, for the $1S_{1/2}$ state, and $(Z\alpha)_{cr} = 1.383$ for $R = 10$ fm, for the $2P_{1/2}$ state, in agreement with Ref. 3. Furthermore, in agreement with Ref. 2, dB/dZ at $Z = Z_{cr}$ was calculated to be 27 keV for the $1S_{1/2}$ state and 35 keV for the $2P_{1/2}$ state. Secondly, the residues at those poles were calculated in order to check that the normalized bound-state wave functions were given correctly. All $S_{1/2}$ and $P_{1/2}$ wave functions calculated thereby were normalized to unity to better than one part in 10^5. The limit $R \to 0$ was taken numerically for $Z\alpha < 1$ to check that these wave functions reduce to the point-nucleus form. In addition, $\langle 1/r \rangle$ was computed for these wave functions since $dB/dZ \approx \alpha m_e c^2 \langle 1/r \rangle$ which can be compared to the values computed above; at $Z = Z_{cr}$, e.g., $dB/dZ = 28$ keV and 37 keV for the $1S_{1/2}$ and $2P_{1/2}$ states, respectively, in good agreement with the above values. Thirdly, for the special case of $Z = 82$, extensive checks on the charge densities were made showing that the limit $R \to 0$ and the finite-nuclear-size effect agreed with other calculations.[6] Finally, the total space integral Q_{He} of ρ_{He} was computed for the range $0.6 \leq Z\alpha \leq 1.38$ since the extent to which $Q_{He} = -2|e|$ is satisfied measures the accuracy of the numerical contour integration along I. The computed values of Q_{He} were equal to $-2|e|$ to better than 1 part in 10^4 over the entire range. The accuracy of the computed ρ^{3+} for $Z < Z_{cr}$ is less as a result of cancelations necessary to insure that the total charge of the vacuum vanishes. This accuracy is estimated from the magnitude of the ratio of the integral of ρ^{3+} over all space ($r \lesssim 13\lambda_e$) to the integral over the range where ρ^{3+} is negative ($r \lesssim 100$ fm). This ratio was found to be $\lesssim 0.01$, indicating a numerical accuracy on the order of one percent.

The computed $1S_{1/2}$ energy shifts for $Z < Z_{cr}$ due to higher-order VP are listed in Table I. These energy shifts should be compared to the shift due to the Uehling potential. The Uehling potential is attractive and increases the binding energy of the $1S_{1/2}$ state by approximately 10 keV at Z_{cr}.[1,10] The higher-order VP is repulsive but is seen in Table I to reduce the binding energy by only 1 keV at Z_{cr}. The last two lines, in particular, in Table I indicate the absence of any singularities of ΔE^{3+} at Z_{cr}. Thus, even though the shift ΔE^{3+} due to higher-order VP increases rapidly for $Z\alpha > 1$, it remains too small to prevent the $1S_{1/2}$ state from reaching the lower continuum. Furthermore, the results for third order indicate that the dependence of ΔE^{3+} on the specific nuclear charge density is a 10% effect, i.e., on the same order as the uncertainty in ΔE^{3+} due to neglect of the $|k| \geq 2$ contribution to the VP density.

Since the results for $Z < Z_{cr}$ show that VP remains a small perturbation up to Z_{cr}, the use of the unperturbed Green's functions, $\mathrm{Tr}G_k$ and $\mathrm{Tr}G_k^1$, in computing $\rho^{3+} \equiv \rho_{He}$ for $Z > Z_{cr}$ will not lead to large errors. In Fig. 2(a), ρ_{He} is plotted for several values of $Z\alpha$ around $(Z\alpha)_{cr} = 1.274\,59$.

TABLE I. The $1S_{1/2}$ energy shifts in eV due to vacuum-polarization orders $\alpha(Z\alpha)^n$, with ΔE^{3+} for $n \geq 3$, and $\Delta E_{I,II}$ for $n=3$, model-I and -II nuclei. $E_{1S_{1/2}}$ and $E_{2P_{1/2}}$ locate the bound-state poles in units of m_e. The nuclear charge density for all but the last column was taken to be model I with $R=10$ fm.

Z	$E_{1S_{1/2}}$	$E_{2P_{1/2}}$	ΔE^{3+}	$\Delta E_{\mathrm{I}}{}^{3}$	$\Delta E_{\mathrm{II}}{}^{3}$
0.95	0.362	0.817	6.26×10	4.70×10	4.92×10
1.12	0.137	0.570	3.07×10^2	1.97×10^2	2.11×10^2
1.205	−0.550	0.265	6.41×10^2	3.68×10^2	3.99×10^2
1.2732	−0.990	−0.118	1.14×10^3	5.66×10^2	6.20×10^2
1.274 45	−0.999	−0.126	1.15×10^3	5.70×10^2	6.24×10^2

The continuity of ρ_{He} at Z_{cr} was examined by calculating ρ_{He} for $Z\alpha=1.2732$, 1.274 45, 1.275 45, and 1.28 and checking point by point in the range $0 \leq r \lesssim 13\lambda\!\!\!^{-}_e$ that the values of ρ_{He} for the different Z can be smoothly connected. The increased localization and continuity of ρ_{He} as a function of Z is illustrated in Fig. 2(b), where the average $\langle 1/r\rangle$ for ρ_{He} is plotted. These results demonstrate that the heliumlike system is stable and well behaved around Z_{cr} and that the charge density of the overcritical vacuum is indeed highly localized.[11]

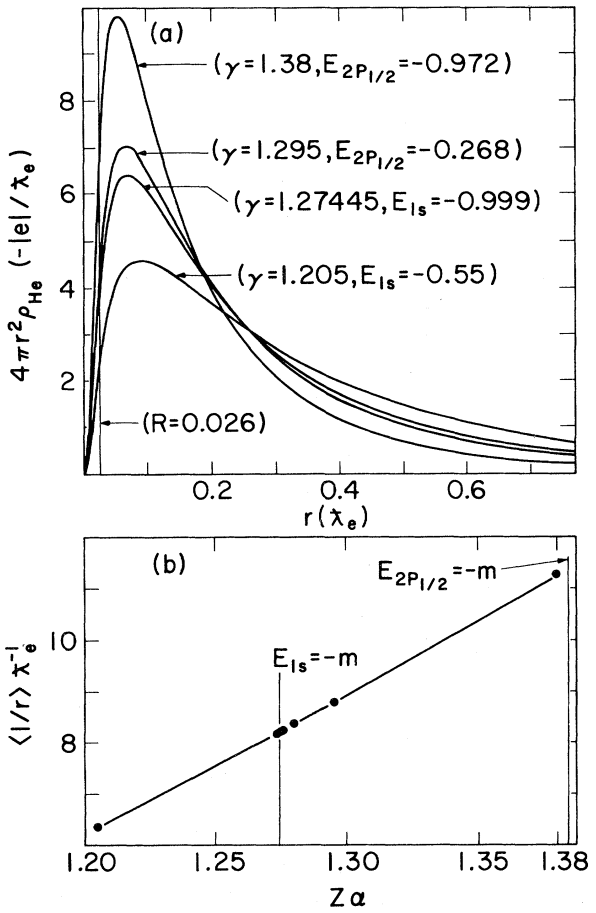

FIG. 2. (a) The heliumlike charge density for several values of $\gamma \equiv Z\alpha$ around $(Z\alpha)_{\mathrm{cr}}=1.274\,59$ with a model-I, $R=10$-fm nucleus. (b) The average $\langle 1/r\rangle$ for ρ_{He} as a function of $Z\alpha$.

It should be noted that for $Z\alpha=1.383$, the $2P_{1/2}$ state reaches the lower continuum and the heliumlike system will decay to a berylliumlike system plus two free positrons.[2,3] The charge density of the berylliumlike system can be studied in the same way as ρ_{He}, simply by shifting the contour C_{He} to the right of the $2P_{1/2}$ pole.

In this Letter the effect of VP in very high-Z atoms has been considered. The effect of the Lamb shift has been estimated by other workers,[10] but agreement on the size of that effect has not yet been reached. More work is needed on that problem.

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General Derivation of Bäcklund Transformations from Inverse Scattering Problems*

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A general way is demonstrated to derive Bäcklund transformations for nonlinear partial differential equations that are solvable by the inverse scattering method in the scheme of Ablowitz, Kaup, Newell, and Segur.

Recently, Ablowitz *et al.*[1] discovered a general scheme for finding the set of nonlinear partial differential equations that are solvable by the inverse scattering method. This paper will show how one can derive the Bäcklund transformation from the auxiliary equations for the inverse problem.[2] This derivation provides the basis for unifying the two different approaches to solving these nonlinear equations.

Ablowitz *et al.*[1] have found that the integrability conditions for the systems of linear partial differential equations

$$v_{1x}+i\zeta v_1=qv_2,\quad v_{2x}-i\zeta v_2=rv_1 \tag{1}$$

and

$$v_{1t}=Av_1+Bv_2,\quad v_{2t}=Cv_1-Av_2 \tag{2}$$

are exactly those equations which allow soliton solutions solvable by the inverse scattering method. The integrability conditions are

$$\begin{aligned}&A_x=qC-rB,\quad B_x+2i\zeta B=q_t-2Aq,\\&C_x-2i\zeta C=r_t+2Ar.\end{aligned} \tag{3}$$

Finite expansions of A, B, and C in terms of ζ reduce the problem to specific equations of interest, for example, Korteweg–de Vries (KdV),[2] modified Korteweg–de Vries (mKdV),[3] sine-Gordon,[4] and nonlinear Schrödinger equations.[5]

Equations that are solvable by an inverse scattering problem are found to be also solvable by Bäcklund transformations.[6,7] But till now only two such transformations have been found.[6,7] They are derived independently from a tedious *ad hoc* elimination procedure (or simply from a guess). I present in this section a unified way of finding them from the inverse problem, and therefore, provide a basis for the statement that corresponding to each inverse problem there exists a Bäcklund transformation.

From Eqs. (1)–(3), we can easily get a system of equations for the quantity $u\equiv v_1/v_2$:

$$u_x=2i\zeta u-ru^2+q,\quad u_t=2Au-Cu^2+B. \tag{4}$$

This equation is very important. I will demonstrate in the following by showing examples that all Bäcklund transformations can indeed be reduced to (or derived from) this set of Riccati equations. A specific identification of u as a functional of q and q' will provide a Bäcklund transformation to a particular differential equation. These Bäcklund transformations can be divided into different classes.

Class I.—For the first class $r=\text{const}=-2$. Equation (4) then becomes, with $i\zeta=k$,

$$u_x=-2ku+2u^2+q,\quad u_t=2Au-Cu^2+B. \tag{5}$$

The simplest example in this class is the Korteweg–de Vries equation,[7] $q_t+12qq_x+q_{xxx}=0$. Following Ablowitz *et al.*, we identify A, B, and C to be

$$\begin{aligned}&A=4k^3+4kq-2q_x,\\&B=-4k^2q+2kq_x-q_{xx}-4q^2,\\&C=8k^2=8q.\end{aligned} \tag{6}$$

If we eliminate q in Eq. (5) we get an equation for u,

$$u_t-24u^2u_x+24kuu_x+u_{xxx}=0. \tag{7}$$

This is a mixed KdV-mKdV equation. Equation (5) provides a Bäcklund transformation between solutions of the KdV equation and Eq. (7). In particular, if $k=0$, Eq. (7) reduces to the pure mKdV equation and Eq. (5) becomes the famous Miura transformation. Now, we can see that if (u,k) satisfies Eq. (7), then $(-u,-k)$ also satisfies