perature) and with the Osheroff and Brinkman results[4] can be considered as satisfactory: Because of the scattering there is no real contradiction. Again a thermal gradient can partially explain the discrepancy.

In the isotropic state, according to Leggett,[2] there is no shift in the transverse resonance in a bulk equilibrium, but there is a longitudinal resonance frequency $\Omega_L(T)$. For the width of this line, we find[15]

$$\Delta\Omega_L = \tfrac{1}{3}\Omega_L{}^2(T)\tau(T)\,\frac{[2f(T)+\varphi(T)]}{\kappa(T)[1+F_0{}^a\kappa(T)]}, \qquad (11)$$

where

$$\kappa(T) = \tfrac{1}{3}[2+\varphi(T)], \qquad (12)$$

and $\varphi(T)$ is the Yoshida function; $f(T)$ is still given by Eq. (10), where the order parameter is now isotropic.

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*On leave from Groupe de Physique des Solides de l'Ecole Normale Supérieure, Université de Paris VII, 75005 Paris, France.

‡On leave from Department of Applied Science, Faculty of Engineering, Tohoku University, Sendai 980, Japan.

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Renormalization-Group Analysis of Bicritical and Tetracritical Points

David R. Nelson, J. M. Kosterlitz,† and Michael E. Fisher‡
Laboratory of Atomic and Solid State Physics, Baker Laboratory of Chemistry, and the Materials Science Center, Cornell University, Ithaca, New York 14850
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Recently developed renormalization-group techniques are summarized and exploited to yield a renormalization-group analysis of bicritical and tetracritical points (which arise in antiferromagnets and boson systems). For $n \lesssim 3$ an isotropic or Heisenberg fixed point dominates and gives bicritical behavior; but for $n \gtrsim 4$ a new "biconical" fixed point with irrational ϵ-expansion coefficients appears. This describes a tetracritical point and may be relevant to displacive phase transitions.

In an earlier note[1] (referred to as I), a scaling theory was developed for bicritical points such as antiferromagnetic spin-flop points and the analogs of the upper λ point in ^{4}He. Here, recently developed renormalization-group techniques[2] are summarized and employed to give concrete numerical predictions for the exponents introduced in I. Three distinct fixed points are found to play a role: As the number of components of the order parameter is varied, either an isotropic Heisenberg, a "biconical," or a "decoupled" fixed point dominates the behavior, which is bicritical in the first case but tetracritical with an intermediate, doubly ordered phase in the second two cases. As in I, we treat a uniaxially anisotropic antiferromagnet of n-component spins $\vec{S}(\vec{R}) = [S_1(\vec{R}) \equiv S_{\parallel}(\vec{R}), \vec{S}_{\perp}(\vec{R})]$ at the sites $\vec{R}$ of a d-dimensional lattice. The interaction Hamiltonian is

$$\mathcal{H}_{int} = -\sum_{\vec{R}\vec{R}'} [(\vec{R} - \vec{R}')\vec{S}(\vec{R}) \cdot \vec{S}(\vec{R}') + D(\vec{R} - \vec{R}')S_{\parallel}(\vec{R})S_{\parallel}(\vec{R}')] - \sum_{\vec{R}} [H_{\parallel}S_{\parallel}(\vec{R}) + \vec{H}_{\perp} \cdot \vec{S}_{\perp}(\vec{R})] - \sum_{\vec{R}} \exp(i\vec{k}_0 \cdot \vec{R})\vec{H}^{\dagger} \cdot \vec{S}(\vec{R}). \tag{1}$$

As discussed in I, $J(\vec{R})$ represents an isotropic exchange coupling, while $D(\vec{R})$ introduces an anisotropy energy aligning the spins along an "easy" or "parallel" axis. The staggered, ordering field is $\vec{H}^{\dagger}$, while $\vec{H} = (H_{\parallel}, \vec{H}_{\perp})$ is a uniform external field. We shall be concerned chiefly with the case $\vec{H}^{\dagger} = 0$ and $\vec{H}_{\perp} = 0$, so that only $H_{\parallel}$ acts. The Hamiltonian (1) will be discussed in fuller detail elsewhere.[3]

Our treatment follows lines developed[2] for discussing layered metamagnets (which display tricritical points[4]). As usual,[5] we take the $\vec{S}(\vec{R})$ to be continuous, classical variables subject to spin weighting factors[5] $e^{-W(\vec{S})}$ with $W(\vec{S}) = \frac{1}{2}|\vec{S}|^2 + f|\vec{S}|^4$. The essential step is then to introduce two spin fields, $\vec{s}_+(\vec{q})$ and $\vec{s}_-(\vec{q})$, via

$$\vec{s}_{\pm}(\vec{q}) = \tfrac{1}{2} \sum_{\vec{R}\subset A} e^{i\vec{q}\cdot\vec{R}}\vec{S}(\vec{R}) \pm \tfrac{1}{2} \sum_{\vec{R}\subset B} e^{i\vec{q}\cdot\vec{R}}\vec{S}(\vec{R}), \tag{2}$$

where A and B denote the two interpenetrating sublattices into which the lattice is decomposed when the spins order antiferromagnetically. The wave vectors $\vec{q}$ run over a *reduced* Brillouin zone corresponding to a superlattice. This construction permits a transparent analysis of the ordering effects associated with competition between two sublattices which might otherwise be missed. The variables $\vec{s}_+(\vec{q})$ and $\vec{s}_-(\vec{q})$ diagonalize the quadratic parts of the reduced Hamiltonian

$$\overline{\mathcal{H}} = -\mathcal{H}_{int}/k_B T - \textstyle\sum_{\vec{R}} W(\vec{S}(\vec{R})). \tag{3}$$

A renormalization-group procedure is then initiated by (i) translating the spin variables in order to eliminate the linear terms; (ii) rescaling spatially by a factor $b > 1$ and integrating out the shifted spin variables $\vec{\sigma}_{\pm}(\vec{q})$ with wave vectors then lying outside the Brillouin zone; (iii) reshifting the spin variables to eliminate the new linear terms generated; and (iv) introducing the distinct spin rescaling factors $\hat{c}_{\parallel}^{+}, \hat{c}_{\perp}^{+}, \hat{c}_{\parallel}^{-}, \hat{c}_{\perp}^{-}$, for the corresponding spin components of $\vec{\sigma}_{\pm}(\vec{q})$. The $\hat{c}^{+}$ factors are chosen to keep the coefficients of $q^2|\sigma_+^{\parallel}(\vec{q})|^2$ and $q^2|\sigma_+^{\perp}(\vec{q})|^2$ in the renormalized Hamiltonian equal to unity, but the $\hat{c}^{-}$ factors are chosen to keep constant the coefficients $r_{\parallel}^{-}$ and $r_{\perp}^{-}$ of $|\sigma_-^{\parallel}(\vec{q})|^2$ and $|\sigma_-^{\perp}(\vec{q})|^2$. Under this previously unexploited rescaling procedure,[2] many terms in $\overline{\mathcal{H}}$ become strongly irrelevant, going rapidly to zero as the renormalization procedure is iterated. In particular, the momentum dependence associated with the $\vec{\sigma}_-(\vec{q})$ spins disappears allowing these variables to be explicitly integrated out of the problem.[6] The resulting reduced, renormalized Hamiltonian contains no terms of odd order in the remaining variables $\vec{\sigma}_+$, and may be written schematically in real space as

$$\overline{\mathcal{H}}_{red} \approx -\tfrac{1}{2}\int d^3R[r_{\parallel}\sigma_{\parallel}^2 + |\nabla\sigma_{\parallel}|^2 + r_{\perp}\vec{\sigma}_{\perp}^2 + |\nabla\vec{\sigma}_{\perp}|^2 + 2u\sigma_{\parallel}^4 + 4w\sigma_{\parallel}^2\vec{\sigma}_{\perp}^2 + 2v\vec{\sigma}_{\perp}^4], \tag{4}$$

where the plus signs and arguments R have been omitted. The quartic coefficients u, v, and w are positive, vary slowly with T and $H_{\parallel}$ (taking $\vec{H}_{\perp} = \vec{H}^{\dagger} \equiv 0$), and satisfy no special relations. The basic $(H_{\parallel}, T)$ variation is found to be

$$r_{\parallel} \approx a_{\parallel}(T - T_{\parallel}) + 12a_0 H_{\parallel}^2, \quad r_{\perp} \approx a_{\perp}(T - T_{\perp}) + 4a_0 H_{\parallel}^2, \tag{5}$$

where, for uniaxial anisotropy [i.e., nonzero $D(\vec{R})$], we have $T_{\parallel} > T_{\perp}$, while a_0, $a_{\parallel}$, and $a_{\perp}$ are positive constants.

The analysis of $\overline{\mathcal{H}}_{red}$ now follows standard lines.[5] For small $H_{\parallel}$, the parameter $r_{\parallel}$ becomes negative before $r_{\perp}$ does, as T is reduced, and the system crosses over to standard Ising-like behavior, as shown by Fisher and Pfeuty.[7] For fields sufficiently large compared to $(T_{\parallel} - T_{\perp})^{1/2}$, the reverse situation occurs, and $(n-1)$-isotropic (i.e., perpendicular or planar) critical behavior is realized.[7] Below T_c this changeover corresponds to the spin-flop transition.

In the case of zero anisotropy [$D(\vec{R}) \equiv 0$], the relations $a_{\parallel} = a_{\perp}$ and $T_{\parallel} = T_{\perp}$ hold, and the bicritical point occurs in zero field. A spin-flop transition to $(n-1)$-isotropic ordering then occurs in any field, however small, since the inequality $r_\perp < r_\parallel$ always holds for $H_\parallel \neq 0$. The associated crossover exponent, $\phi = \phi_H(n)$, is just that discussed by Pfeuty, Jasnow, and Fisher[8] for anisotropic spin systems.

Assuming finite anisotropy $[D(\vec{R}) \neq 0]$, a new analysis is needed in the bicritical region where $r_\parallel \simeq r_\perp$. By assuming that u, v, w, $r_\parallel$, and $r_\perp$ are all of order $\epsilon = 4-d$, recursion relations are readily constructed to the leading order[5,7] and found to be

$$r_\parallel{}' = b^2[r_\parallel + 12fu + 4f(n-1)w - 12gur_\parallel - 4g(n-1)wr_\perp], \tag{6}$$

$$r_\perp{}' = b^2[r_\perp + 4f(n+1)v + 4fw - 4g(n+1)vr_\perp - 4gwr_\parallel], \tag{7}$$

$$u' = b^\epsilon[u - 36gu^2 - 4g(n-1)w^2], \tag{8}$$

$$v' = b^\epsilon[v - 4g(n+7)v^2 - 4gw^2], \tag{9}$$

$$w' = b^\epsilon w[1 - 12gu - 4g(n+1)v - 16gw], \tag{10}$$

where $f(b) = \Lambda^2(b^{-2}-1)/8\pi^2$ and $g(b) = \ln b/8\pi^2\Lambda^\epsilon$ arise from the usual Feynman-type integrals over the outer momentum shell with cutoff Λ, evaluated as $d \to 4$.

For any value of n (> 0) the last three of these equations determine *six* fixed points. Four of these have $w^* = 0$, and so represent *decoupled* and, hence, *tetracritical* Hamiltonians: (a) $u^* = v^* = 0$ is the trivial, always unstable, Gaussian-Gaussian point; putting $\bar{\epsilon} = 8\pi^2\Lambda^\epsilon\epsilon$ we have (b) $u^* = \bar{\epsilon}/36$, $v^* = 0$ which is an Ising-Gaussian point; similarly, (c) $u^* = 0$, $v^* = \bar{\epsilon}/4(n+7)$, is a Gaussian-$(n-1)$-Heisenberg point; lastly (d) $u^* = \bar{\epsilon}/36$ $v^* = \bar{\epsilon}/4(n+7)$ is a decoupled-Ising-$(n-1)$-Heisenberg fixed point. For $n < 11 + O(\epsilon)$ all these fixed points are found to be unstable to the w perturbations.

Of the two remaining fixed points the first, which describes *bicritical behavior*, (e) is located at

$$u^* = w^* = v^* = \bar{\epsilon}/4(n+8)$$

with

$$r_\parallel = r_\perp = -\bar{\epsilon}(n+2)/2(n+8). \tag{11}$$

This is easily recognized as the fully isotropic n-Heisenberg fixed point,[5,7] thus confirming the statement reported in I and exploited there to make numerical predictions. This fixed point remains stable[7b,9] in the full (u, v, w) subspace for $n < n^\times(d) = 4 - 2\epsilon + c^\times\epsilon^2 + O(\epsilon^3)$, where[10] $c^\times = \frac{5}{12}[6\zeta(3) - 1]$. To evaluate this at $d = 3$, a Padé approximant to the series has been formed, with the result $n^\times(d) \approx (4 + 3.176\epsilon)/(1 + 1.294\epsilon) \simeq 3.128$ at $\epsilon = 1$. The fact that both $r_\parallel$ and $r_\perp$ are specified at the fixed point indicates that bicriticality is attained only at isolated points in the $(H_\parallel, T)$ plane, as anticipated.[1] All the exponents associated with this fixed point (including the crossover exponent ϕ) are just those of the usual n-isotropic, Heisenberg model.

For $n^\times(d) < n < 11 + O(\epsilon)$, a new fixed point (f) finally becomes stable: This is located at

$$w^* = \bar{\epsilon}x/8, \quad u^* = \{1 + [1 - 9(n-1)x^2]^{1/2}\}\bar{\epsilon}/72, \quad v^* = \{1 + [1 - (n+7)x^2]^{1/2}\}\bar{\epsilon}/8(n+7), \tag{12}$$

$$r_\parallel{}^* = [12fu^* + 4(n-1)fw^*]/(b^{-2}-1), \quad r_\perp{}^* = [4(n+1)fv^* + 4fw^*]/(b^{-2}-1), \tag{13}$$

where x is the real root of

$$9(4n^2 + 29n + 88)x^3 - 6(2n^2 + 28n + 179)x^2 + (n^2 + 5n + 472)x + 6(n-11) = 0. \tag{14}$$

Although the appropriate root of this equation is rational at $n = 11$ $(x = 0)$, $n = 4$ $(x = \frac{1}{6})$, $n = 2$ $(x = \frac{1}{3})$, and at $n = 1$ and -1, the root is an irrational function of n; specifically, for $n = 5$, we have $x = [82 - (a + b\sqrt{82})^{1/3} - (a - b\sqrt{82})^{1/3}]/333$, where $a = 18\,728$ and $b = 1998$. The renormalization-group eigenvalues, and thence the critical-point exponents, can be calculated to order ϵ through (6) to (10) and again have irrational coefficients. There is, for example, a cube-root *cusp* in the corresponding susceptibility exponent $\gamma_B(b)$ at $n = 2$ as shown in Fig. 1.

In the region of stability it is not hard to show that $0 \leq w^* < \bar{\epsilon}/(n+8)$ while u^* and v^* exceed $\bar{\epsilon}/(n+8)$. Accordingly this fixed point then satisfies the condition $(w^*)^2 < u^*v^*$ which represents the phenomenological criterion[11] for *tetracritical-*

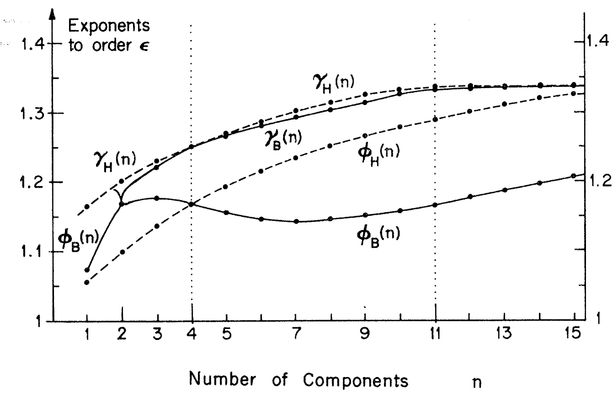

FIG. 1. Comparison of the exponents γ and φ at the new, biconical fixed point (solid curve) and the isotropic (Heisenberg) fixed point (dashed line) truncated at order ϵ and evaluated at $\epsilon = 1$. Note that $\gamma_B(n)$ has a cusp at $n=2$.

ity, i.e., a new intermediate phase with *both* parallel and perpendicular order simultaneously present is thus expected to appear below T_b as illustrated in I. Recall that the equation of state to order ϵ^0 is always given by the phenomenological theory, and we do not expect the corrections of order ϵ and higher to alter such qualitative features of the fixed point. [Note that the condition for *bi*criticality,[11] namely, $(w^*)^2 \geq u^*v^*$, is satisfied at the Heisenberg critical point so that, within the scaling regime, tetracritical behavior should not be realizable for $n < n^{\times}(d)$.]

Because of the symmetry implied by the unequal values of $r_{\parallel}^*$ and $r_{\perp}^*$, and by the values of u^*, v^*, and w^*, we call this new fixed point "biconical." Thus the spins tend to lie on an "easy cone" with axis parallel to the original easy axis, and with a conical angle determined by n (via the fixed-point values).

The new, biconical susceptibility and crossover exponents, $\gamma_B(n)$ and $\phi_B(n)$, are plotted versus n in Fig. 1 according to the truncated expansions evaluated at $\epsilon = 1$. The corresponding, truncated values of the n-isotropic exponents $\gamma_H(n)$ and $\phi_H(n)$ are shown for comparison. In fact, the value for $\gamma_B(n)$ falls only 0.01 or less below $\gamma_H(n)$ (at $\epsilon = 1$). [Likewise the values for $\alpha_B(n)$ are only slightly less negative than for $\alpha_H(n)$.] However, the crossover exponent $\phi_B(n)$, to order ϵ, stays almost constant for $5 \leq n \leq 11$; by contrast $\phi_H(n)$ rises quite rapidly towards $\gamma_H(n)$ as n increases.

For $n > 11 + O(\epsilon)$, the decoupled fixed point (d) dominates. The Hamiltonian spontaneously breaks into independent Ising and $(n-1)$-Heisenberg systems. As observed in I a single scaling function cannot properly describe this fixed point. Rather, the singular part of the free energy is the sum of two distinct scaling functions. This mechanism for the breakdown of scaling is a striking novel feature of the renormalization group.

To obtain a fuller understanding of these anisotropic systems, we have generalized the Hamiltonians (1) and (4) to include $m_{\parallel}$ parallel and $m_{\perp} = n - m_{\parallel}$ perpendicular spin components.[12] The results,[3] to order ϵ, show that the Heisenberg fixed point dominates, as expected, for $m_{\perp} + m_{\parallel} < 4 + O(\epsilon)$. If the relation

$$m_{\parallel} m_{\perp} + 2(m_{\parallel} + m_{\perp}) > 32 \qquad (15)$$

holds, the decoupled fixed point dominates. The biconical fixed point determines the (tetracritical) behavior in the remaining region of the $(m_{\parallel}, m_{\perp})$ plane. While these biconical and decoupled fixed points are probably irrelevant to simple magnetic and boson systems they may well play a significant role in the description of polycritical points in displacive transitions, where crystal symmetries may introduce order parameters of higher dimensionality.

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†Permanent address: Department of Mathematical Physics, The University, Birmingham, England.

‡To whom reprint requests should be addressed at Baker Laboratory.

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