Eq. (3). The latter includes consecutive decay in ions with multiple vacancies in the inner shell and metastable decays which in some cases may contribute to N_a but not to N_i.

Condition Eq. (5) holds for K x rays of sulfur ions with energies above ~30 MeV, mainly because of relatively large electron-capture cross sections σ_c. It must be expected that Eq. (5) is validated when the ion velocity reaches an appreciable fraction of the orbital velocity associated with the projectile vacancy. We estimate, for example, that specific sulfur L x-ray satellites should exhibit a nonproportional yield curve at energies above a few hundred keV. Finally, we note that the phenomenon of nonproportionality between projectile x-ray yield and target thickness affects measurement of projectile x-ray production cross sections and allows determination of the number of ions which emerge from solids in certain states of excitation.

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NMR Shift and NMR Linewidth in Superfluid ^{3}He†

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A set of equations is given, which allows us to rederive both the results of Leggett (hydrodynamic regime) and of Maki and Ebisawa (collisionless regime) for NMR in superfluid ^{3}He. The NMR linewidths for both transverse and longitudinal resonance are derived by introducing a relaxation-time approximation. The results are in good agreement with experimental data.

NMR has been recently shown to be a very powerful probe of the superfluid phases of liquid ^{3}He. The most striking feature is the shift from the Larmor frequency in transverse-resonance experiments.[1] Leggett has given a theory[2,3] accounting for this shift very well. Moreover, his prediction for a longitudinal resonance has been confirmed recently by experiment,[4,5] strongly indicating that the A phase is the axial state, in agreement with the theory by Anderson and Brinkman.[6] Recent NMR experiments[4] also support the hypothesis that the B phase is the isotropic state first considered by Balian and Werthamer.[7]

In this Letter, we rederive Leggett's results in the limit of small fluctuations and obtain linewidths of the NMR lines. This is done by using a completely microscopic theory which enables us to treat both the collisionless and the hydrodynamic regime. In this way, the results of Maki and Ebisawa[8,9] (collisionless regime) and the results of Leggett[2,3] (hydrodynamic regime) are deduced from the same set of equations. Moreover, we take into account collisions between quasiparticles by introducing a relaxation time in our equations. We obtain in the axial state both transverse and longitudinal linewidths which are in good agreement with the existing experimental data.[4,5] The surprisingly large resulting linewidths, induced by quasiparticle collisions, have the same physical origin as the NMR shift, namely the symmetry-breaking effect of the dipole interaction. We also give the result for the longitudinal linewidth in the isotropic state, but there are no published data on the B phase.

The method that we use is based on the kinetic equation for the distribution function, which is derived by using the Hartree-Fock approximation from a Hamiltonian that includes pairing, Fermi liquid, and Zeeman terms in addition to kinetic energy. This equation is expanded in lowest order in ω/q and qv_F/Δ and diagonalized. The small fluctuations of the direction of the order parameter are accounted for by a space-time-dependent rotation in spin space, $U=\exp[-i\vec{\sigma}\cdot\vec{\theta}(\vec{r},t)]$. This method has been described extensively in another paper[10] and used to study spin waves in

the collisionless and hydrodynamic regime with and without magnetic field.[10,11]

In the present case, we are interested only in space-independent fluctuations. The kinetic equation turns out to be

$$\omega\delta\vec{\nu}_k - 2i\delta\vec{E}^0\times\delta\vec{\nu}_k = -i\delta\tilde{\vec{\nu}}_k/\tau, \tag{1}$$

where we have just introduced a collision-time approximation on the right-hand side. The second term, on the left-hand side, is a Larmor term. In this equation, $\delta\vec{\nu}_k$ is the spin part of the fluctuations of the quasiparticle distribution of momentum $\vec{k}$ and $\delta\tilde{\vec{\nu}}_k$ is its departure from local equilibrium. These fluctuations are related to each other by

$$\delta\tilde{\vec{\nu}}_k = \delta\vec{\nu}_k - \varphi'\delta\vec{E}_k; \quad \varphi = -\tfrac{1}{2}\tanh(\beta E_k/2); \quad \varphi' = \partial\varphi/\partial E_k, \tag{2}$$

where $\delta\vec{E}_k$ is the local change in the spin part of the energy matrix[10,11] and is related to $\vec{\theta}$ and the magnetization by

$$\delta\vec{E}_k = \frac{\xi_k}{E_k}\vec{X} + \left(1-\frac{\xi_k}{E_k}\right)\frac{\vec{d}_k}{|\Delta_k|^2}(\vec{d}_k^{\,*}\cdot\vec{X}); \tag{3}$$

$$\vec{X} = -i\omega\vec{\theta} + \vec{\omega}_L\times\vec{\theta} + f_0^{\,a}\delta\tilde{\vec{\rho}}; \quad \vec{\omega}_L = \gamma\vec{H}; \tag{4}$$

$$\delta\tilde{\vec{\rho}} = \delta\vec{\rho} + 2\vec{\theta}\times\vec{\rho}_0 = \sum_k\left\{\frac{\xi_k}{E_k}\delta\vec{\nu}_k + \left(1-\frac{\xi_k}{E_k}\right)\frac{\vec{d}_k}{|\Delta_k|^2}(\vec{d}_k^{\,*}\cdot\delta\vec{\nu}_k) + \frac{\varphi}{E_k^{\,3}}\vec{d}_k\times(\vec{X}\times\vec{d}_k^{\,*})\right\}; \tag{5}$$

where $\xi_k = \hbar^2k^2/2m^* - \mu$, $E_k = (\xi_k^{\,2} + |\Delta_k|^2)^{1/2}$, $\vec{\rho}_0$ is the static magnetization, and $\delta\vec{\rho}$ is the fluctuation of the magnetization. In these equations, $\vec{d}_k$ is related to the order-parameter matrix by $\Delta_k = i(\vec{\sigma}\cdot\vec{d}_k)\sigma_y$ and only the s-wave part of the antisymmetric Fermi-liquid parameters, $F_0^{\,a} = N_0 f_0^{\,a}$, is retained.

To complete the set of equations, we need the equation of motion for the magnetization $\delta\vec{\rho}$. Here we have to take the dipole-dipole interaction into account which we have neglected so far because, as pointed out by Leggett,[2,3] the inclusion is unimportant except in the equation for $\delta\vec{\rho}$. The equation is derived by taking the commutator of $\delta\vec{\rho}$ and the Hamiltonian, including the dipole term (treated within the Hartree-Fock approximation). Then, by performing the time-dependent rotation $\vec{\theta}(t)$ and ignoring ω/Δ terms, we obtain

$$\omega\delta\vec{\rho} + i\vec{\omega}_L\times\delta\vec{\rho} - i\underline{\Phi}^{(0)}\vec{\theta} = 0. \tag{6}$$

The third term comes from the dipole interaction, where $\underline{\Phi}^{(0)}$ is the tensor introduced by Leggett.[2]

From the kinetic equation (1) and the self-consistent equations (3), (4), and (5), we can solve for $\delta\vec{\rho}$ in terms of $\vec{\theta}$. Then, together with the equation of motion (6), we can study the continuous-wave NMR in any regime.

In the collisionless regime, we have rederived, in our low-frequency limit $\omega/\Delta \ll 1$, the results of Maki and Ebisawa[8,9] for both the longitudinal and the transverse resonance in the axial state [in that case $2\delta\vec{E}^0 = -\xi\vec{\omega}_L/E(1+F_0^{\,a})$]. However, our present interest is in the hydrodynamic regime $\omega\tau \ll 1$, where the experiments are actually done. To first order in $\omega\tau$, Eq. (1) gives

$$\delta\vec{\nu}_k = (1+i\omega\tau)\varphi'\delta\vec{E}_k. \tag{7}$$

In the limit $\omega\tau \to 0$, we recover $\delta\vec{\nu}_k = \varphi'\delta\vec{E}_k$ (local equilibrium) which then gives with Eqs. (3), (4), and (5)

$$\delta\tilde{\vec{\rho}} = \underline{\chi}(i\omega\vec{\theta} - \vec{\omega}_L\times\vec{\theta}) \tag{8}$$

where $\underline{\chi}$ is the static susceptibility tensor with fixed $\vec{d}_k$. If $\underline{\chi}$ is replaced by an isotropic χ, as done by Leggett, Eq. (6) together with Eq. (8) reduces to Leggett's Eq. (12) of Ref. 2 (in our notation $2\vec{\rho}_0 = \underline{\chi}\vec{\omega}_L$). Here we have not included the rf field but this could have been done very easily.

The deviation from local equilibrium of order $\omega\tau$ leads to the linewidth. We look first at the axial state, retaining the anisotropic susceptibility tensor, although an isotropic one leads to the same result. Taking twice the imaginary part of the frequency, we obtain the full widths for transverse and longitudinal resonances:

$$\Delta\omega_T = \frac{\tau(T)}{1+F_0^{\,a}}\frac{\omega_0^{\,4}(T)}{\omega^2}f(T), \qquad \Delta\omega_L = \omega_0^{\,2}(T)\tau(T)\frac{f(T)}{1+F_0^{\,a}}, \tag{9}$$

respectively, where $\omega_0(T)$ is the frequency shift: $\omega_0^{\,2}(T) = \omega^2 - \omega_L^{\,2}$ and

$$f(T) = \int\frac{d\Omega_k}{4\pi}\int_{-\infty}^{+\infty}d\xi(-\varphi')\left(\frac{\xi}{E}\right)^2. \tag{10}$$

To compare this result with experiment, we need to estimate $\tau(T)$. We keep the $1/T^2$ dependence of the normal state and take the magnitude

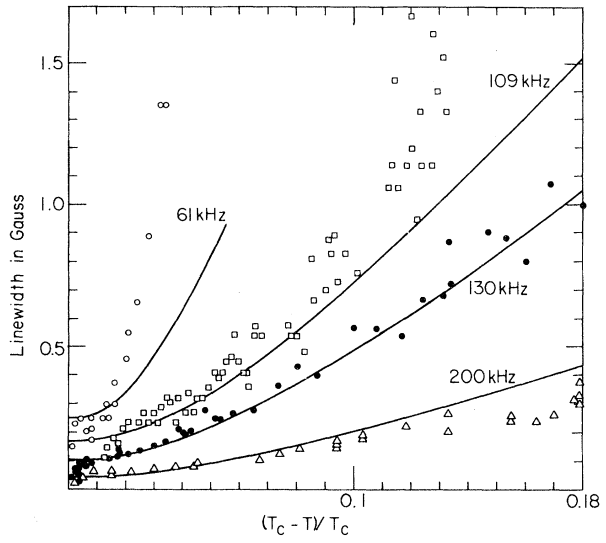

FIG. 1. NMR linewidth for transverse resonance in the A phase. Full lines: theory. Experimental data are from Ref. 5.

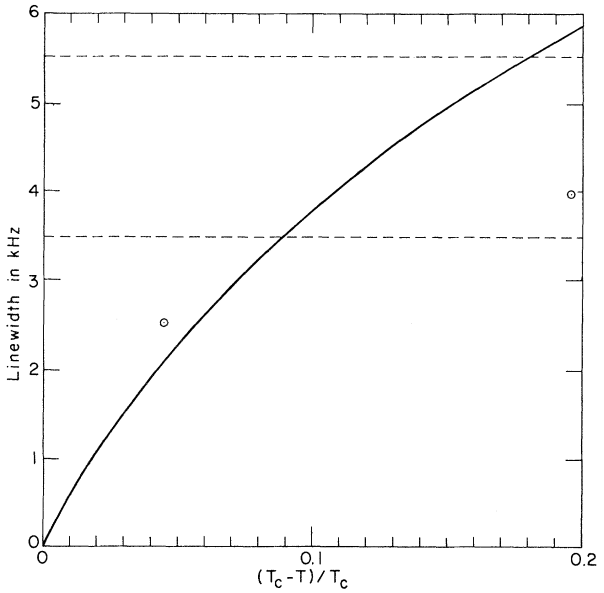

FIG. 2. NMR linewidth for longitudinal resonance in the A phase. Full line: theory. The range of the results from Ref. 14 is between the dashed lines. The open circles are data from Ref. 4.

of τ at T_c to obtain agreement with experimental widths. The change of $\tau(T)$ due to superfluidity should be unimportant because the inelastic character of the processes involved in the quasiparticle collisions will wipe out most of the effect of the gap, at least in the temperature range where experiments are performed. In any case, our result is not very sensitive to $\tau(T)$ in this temperature range, provided that it is a smooth function of T.

In Fig. 1, we have plotted $\Delta\omega_T$ from Eq. (9), with $\tau(T_c) = 0.33\times10^{-7}$ sec and $F_0^a = -0.75$,[12] and experimental results from Ref. 5. $\omega_0(T)$ is taken from experiment and $f(T)$ calculated from the weak-coupling (BCS) theory. The value of our parameter $\tau(T_c)$ is fairly close to the spin-diffusion relaxation time τ_D at T_c, for 27 atm, given by Wheatley[13] ($\tau_D T^2 = 2.1\times10^{-13}$ sec K^2). In a comparison with Eq. (9), an apparent discrepancy was that the experimental width did not extrapolate to zero at T_c, unless the reduced temperature is within 5×10^{-3} of T_c. This can be explained qualitatively by assuming a temperature fluctuation (due, for example, to a thermal gradient) of order 2 μK, which gives, through $\omega_0^2(T)$, a temperature-independent additional width, except very near T_c where it drops to zero. This width depends on ω like $1/\omega$ which roughly agrees with the extrapolated experimental width at T_c. We have added to our theoretical result Eq. (9) a temperature-independent width equal, at each frequency, to this experimental width extrapolated at T_c.

The temperature dependence agrees very well with experiment, except for the 61-kHz data. It is parabolic near T_c, because of $\omega_0^4(T)$; but, because $f(T)$ is dropping very rapidly with the temperature, one obtains a roughly linear dependence in the rest of the experimental temperature range. The $1/\omega^2$ dependence on the frequency is rather well obeyed although the experiment gives a slightly stronger dependence. Although the 61-kHz data have the largest scatter, especially at lower temperature, they are clearly at variance with our result. We note also that the extrapolated width for zero field is far greater than the experimental longitudinal width. The same feature appears at low temperature for the 109-kHz data. This is in contradiction with our result Eq. (9) and seems to indicate that there are additional causes for broadening at low frequency and low field, for example, domain effects.

In Fig. 2, we have plotted $\Delta\omega_L$ from Eq. (9) with the same parameters as for $\Delta\omega_T$. The experimental data on the longitudinal linewidth are very scattered, which makes the comparison with our theory difficult. In these conditions the agreement that we obtained with the Bernier *et. al.*[14] results (4.5 ± 1 kHz, independent of the tem-

perature) and with the Osheroff and Brinkman results[4] can be considered as satisfactory: Because of the scattering there is no real contradiction. Again a thermal gradient can partially explain the discrepancy.

In the isotropic state, according to Leggett,[2] there is no shift in the transverse resonance in a bulk equilibrium, but there is a longitudinal resonance frequency $\Omega_L(T)$. For the width of this line, we find[15]

$$\Delta\Omega_L = \tfrac{1}{3}\Omega_L{}^2(T)\tau(T)\frac{[2f(T)+\varphi(T)]}{\kappa(T)[1+F_0{}^a\kappa(T)]}, \qquad (11)$$

where

$$\kappa(T) = \tfrac{1}{3}[2+\varphi(T)], \qquad (12)$$

and $\varphi(T)$ is the Yoshida function; $f(T)$ is still given by Eq. (10), where the order parameter is now isotropic.

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Renormalization-Group Analysis of Bicritical and Tetracritical Points

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Recently developed renormalization-group techniques are summarized and exploited to yield a renormalization-group analysis of bicritical and tetracritical points (which arise in antiferromagnets and boson systems). For $n \lesssim 3$ an isotropic or Heisenberg fixed point dominates and gives bicritical behavior; but for $n \gtrsim 4$ a new "biconical" fixed point with irrational ϵ-expansion coefficients appears. This describes a tetracritical point and may be relevant to displacive phase transitions.

In an earlier note[1] (referred to as I), a scaling theory was developed for bicritical points such as antiferromagnetic spin-flop points and the analogs of the upper λ point in ^{4}He. Here, recently developed renormalization-group techniques[2] are summarized and employed to give concrete numerical predictions for the exponents introduced in I. Three distinct fixed points are found to play a role: As the number of components of the order parameter is varied, either an isotropic Heisenberg, a "biconical," or a "decoupled" fixed point dominates the behavior, which is bicritical in the first case but tetracritical with an intermediate, doubly ordered phase