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Approximate Renormalization Group Based on the Wegner-Houghton Differential Generator*

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We give an approximate renormalization-group formulation which parallels that of Wilson. The group generator represents the momentum-independent limit of the differential generator of Wegner and Houghton. The eigenfunctions near the Gaussian point are computed for all spin dimensions n and lattice dimensions d, including $d=2$. The nontrivial fixed-point Hamiltonian in dimensions near $d=2\mathcal{O}/(\mathcal{O}-1)$, together with the eigenvalues near that nontrivial fixed point, are found explicitly to first order in $\epsilon_{\mathcal{O}} \equiv \mathcal{O}(2-d)+d$ for *all* values of n and the order $\mathcal{O}$. Odd-dominated Ising systems and corresponding expansions in $\epsilon_{\mathcal{O}-1/2}$ are also treated.

The renormalization-group approach to the study of critical phenomena has had great initial success.[1-2] The renormalization group embodies in concrete mathematical form the scaling notions of Kadanoff[3] and provides a framework for explicit calculation. These calculations have usually been done by perturbative expansions, in analogy with similar problems in quantum field theory. All the difficulties of field theory have been incorporated into critical-phenomena calculations as well; the calculation of thermodynamic quantities involves complicated Feynman diagrams and divergent integrals.

Even in those cases where field-theoretic difficulties are not encountered, the perturbation techniques have been "brute force" in nature. For example, the calculation of critical-point exponents for higher-order[4] critical points has been hampered by the rapid increase of the number of equations which contribute.[5]

Many renormalization-group problems can be simplified by revising the perturbative techniques to conform as closely as possible to the structure of the renormalization group itself. It was noted by Wegner[6] that the eigenfunctions of Wilson's approximate renormalization group (when linearized around the Gaussian point[2]) are related to Laguerre polynomials. However, this observation has hitherto not been fully exploited. Here we show that by utilizing the structure of the renormalization group, a number of problems [see (i)–(iv) below] may be solved simply and explicitly.

To do this, we first write down an appropriate differential equation based upon the Wegner-Houghton[7] differential generator for the renormalization group. Their functional integrodifferential equations may be simplified if we consider them in the limit of vanishing "external" momenta.[2] We find that for n-dimensional isotropically interacting spins $\vec{s}$ on a d-dimensional lattice, the renormalization action on the reduced Hamiltonian H is given by

$$\dot{H} = dH + (2-d)x\frac{\partial H}{\partial x} + \frac{d}{2}\left[1 - \frac{1}{n}\ln\left(1+\frac{\partial H}{\partial x}\right) + \frac{1}{n}\ln\left(1+\frac{\partial H}{\partial x}+2x\frac{\partial^2 H}{\partial x^2}\right)\right], \tag{1}$$

where the dot denotes differentiation with respect to the renormalization parameter l, and $x \equiv (\vec{s}\cdot\vec{s})/n$.[8] Since we have neglected the detailed momentum dependence in the renormalization group, we have set $\eta = 0$.

(i) *The general $\epsilon_{\mathcal{O}}$ expansion.*—To solve (1), the Hamiltonian H can be expanded in terms of any complete set of functions; the expansion functions should be chosen to simplify the problem under consideration. A particularly useful set of functions are the eigenfunctions of (1) when (1) is linearized about the Gaussian fixed point, $H = 0$. These functions (not normalized) can be chosen to be

$$Q_p(x) \equiv [d/(2-d)n]^p L_p^{n/2-1}([(d-2)/d]nx), \qquad (2)$$

where the conventions of Erdelyi[9] are used for the Laguerre polynomials, $L_p^{n/2-1}(z)$. The eigenvalue corresponding to Q_p is $\lambda_p = p(2-d)+d$. To illustrate the use of the Q_p, we have calculated the nontrivial fixed-point Hamiltonians, $H = H_{\mathcal{O}}^*$, corresponding to critical points of order $\mathcal{O}$.[4] The fixed points of (1) are determined by setting $\dot{H} = 0$. In analogy with the ϵ expansions introduced in Refs. 1 and 2, we calculate $H_{\mathcal{O}}^*$ as a perturbation expansion in $\epsilon_{\mathcal{O}} \equiv \mathcal{O}(2-d)+d$,[10] for $\mathcal{O} = 2, 3, 4, \ldots$ (the usual[1,2] ϵ is ϵ_2 in our notation). To first order in $\epsilon_{\mathcal{O}}$, $H_{\mathcal{O}}^* = \epsilon_{\mathcal{O}} v_{\mathcal{O}} Q_{\mathcal{O}}$, where $v_{\mathcal{O}}$ is given by

$$1 = \tfrac{1}{4} d v_{\mathcal{O}} \langle \mathfrak{D}(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle. \qquad (3a)$$

Here the bilinear functional $\mathfrak{D}(i,j)$ is given by

$$\mathfrak{D}(i,j) \equiv \left(1-\frac{1}{n}\right)\frac{dQ_i}{dx}\frac{dQ_j}{dx} + \frac{1}{n}\left((1-n)\frac{dQ_i}{dx} + (2i+n-2)Q_{i-1}\right)\left((1-n)\frac{dQ_j}{dx} + (2j+n-2)Q_{j-1}\right), \qquad (3b)$$

and the inner product $\langle f | p \rangle$ for a function $f(x)$ is defined by

$$f(x) = \sum_{p=0}^{\infty} \langle f | p \rangle Q_p(x). \qquad (3c)$$

Equation (1) can now be linearized around $H_{\mathcal{O}}^*$. The eigenfunctions will change slightly and so will the eigenvalues. If we denote by $\hat{\lambda}_l$ the eigenvalue of the new eigenfunction, which to zeroth order is Q_l, we find that to first order in $\epsilon_{\mathcal{O}}$

$$\hat{\lambda}_l = \lambda_l - 2\epsilon_{\mathcal{O}} \frac{\langle \mathfrak{D}(\mathcal{O}, l) | l \rangle}{\langle \mathfrak{D}(\mathcal{O},\mathcal{O}) | \mathcal{O} \rangle}. \qquad (4)$$

The evaluation of the bilinear coefficients in (4) is merely a problem in classical analysis. In fact, using the full renormalization-group equations, we have shown that (4) is exactly correct[11] to order $\epsilon_{\mathcal{O}}$.

For $n = 1$ (Ising systems), (3b) simplifies considerably, the Q_p are related to Hermite polynomials, and (4) reduces to

$$\hat{\lambda}_l = [l(2-d)+d] - 2\epsilon_{\mathcal{O}}\left(\frac{(2l)!}{(2\mathcal{O})!}\frac{\mathcal{O}!}{(2l-\mathcal{O})!}\right). \qquad (5)$$

These results are in agreement with the $\mathcal{O} = 2$ calculations of Refs. 1 and 2, and the $\mathcal{O} = 3, 4$ calculations of Ref. 5. We note that (5) also contains the odd eigenvalues for $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

From (5) we immediately deduce several important consequences. (i) For $\epsilon_{\mathcal{O}} > 0$, the correction to the Gaussian eigenvalue is negative, so that the nontrivial fixed point always dominates the Gaussian fixed point sufficiently near the critical point. (ii) The correction to the Gaussian eigenvalue vanishes unless $2l \geq \mathcal{O}$. In particular, to order $\epsilon_{\mathcal{O}}$, $\hat{\lambda}_1 = 2$ for all $\mathcal{O} \neq 2$, independent of d. (iii) We note that $\hat{\lambda}_{\mathcal{O}} = -\lambda_{\mathcal{O}} = -\epsilon_{\mathcal{O}}$, so that if we examine the first $\mathcal{O}$ eigenvalues we find that at the Gaussian fixed point they are all positive, and at the nontrivial fixed point all but the last remain positive. The Gaussian point is unstable, and the nontrivial point is a generalized saddle point for $\epsilon_{\mathcal{O}} > 0$.[12]

We also note that the ordering field which couples directly to $\vec{s}$ is entirely decoupled from the remainder of the renormalization-group transformations.[13] The eigenvalue $\lambda_{1/2}$, corresponding to the ordering field, is exactly $1 + d/2$.

(ii) *Gaussian eigenfunctions for $d = 2$.*—We next consider the behavior of (1) for $d = 2$. The nontrivial fixed points at $d = 2\mathcal{O}/(\mathcal{O}-1)$ cluster densely around $d = 2$ as $\mathcal{O} \to \infty$. By studying (1) with d set equal to 2 [or by examining the limit of (2) as $d \to 2$ with $p(2-d)$ fixed] we find the eigenfunctions around the Gaussian fixed point have a continuous set of eigenvalues, $\lambda \leq 2$. A complete orthonormal set of eigenfunctions is given by[14]

$$Q_\lambda(x) = (\tfrac{1}{2}n)^{1/2} x^{-(n/2-1)/2} J_{n/2-1}((4-2\lambda)^{1/2}(nx)^{1/2}), \qquad (6a)$$

where $J_{n/2-1}$ denotes the Bessel function of the first kind, and

$$\int_0^\infty dx\, x^{n/2-1} Q_\lambda(x) Q_{\lambda'}(x) = \delta(\lambda - \lambda'). \qquad (6b)$$

The Hamiltonian is expressible as an integral, $H=\int v_\lambda Q_\lambda d\lambda$, rather than a sum (for $d\neq 2$). In the discrete case, thermodynamic potentials are generalized homogeneous functions[15] of the expansion coefficients. In the continuum limit, they become generalized homogeneous functionals with similar properties. For example, the Gibbs potential satisfies

$$e^{dl}G(v_\lambda)=G(e^{\lambda l}v_\lambda). \tag{7}$$

The continuous nature of the eigenvalue spectrum leads, in general, to logarithmic factors multiplying the usual power-law dependence of generalized homogeneous functions.[15] Since the approximations made in deriving (1) require setting $\eta=0$ for consistency, one must be cautious in interpreting our results for $d=2$.

(iii) *Power-law expansions.*—The solution of (1) for other than ϵ_Θ expansions is more difficult. For n arbitrary, the expansion of H in terms of Laguerre polynomials leads to equations coupled to all orders in the expansion parameters. If these cannot be assumed small, the equations are too complicated for immediate solution. If, however, H is expanded in powers of x, the resulting equations, while not appropriate for general ϵ_Θ analysis, are essentially "triangular." That is, if we expand

$$H=\sum_{j=0}^{\infty}v_{2j}x^j/j\,!,$$

the generator for the v_{2p} equation is given by

$$\dot{v}_{2p}=[p(2-d)+d]v_{2p}+\frac{d}{2}\left(\frac{\partial}{\partial x}\right)^p\left[\left(1-\frac{1}{n}\right)\ln\left(1+\sum_{j=0}^{\infty}\frac{v_{2j}x^{j-1}}{(j-1)!}\right)+\frac{1}{n}\ln\left(1+\sum_{j=0}^{\infty}\frac{(2j-1)v_{2j}x^{j-1}}{(j-1)!}\right)\right]\Bigg|_{x=0}. \tag{8}$$

The linear structure has only one off-diagonal term, $d(1+2p/n)\hat{v}_{2p+2}/2$, and the nonlinear terms are at most of order p in the modified coupling constants $\hat{v}_{2j}=v_{2j}/(1+v_2)$. Furthermore, the nonlinear terms include no v_{2j} with $j>p$. In particular, for $n=-2m$, the first m equations decouple entirely from the remaining equations.[16]

We have used (8) to evaluate critical-point exponents for the ordinary and tricritical points ($\Theta=2,3$). For $\Theta=2$, our results agree with those of Refs. 1 and 2. For $\Theta=3$ we find to order ϵ_3,

$$\hat{\lambda}_1=2,\quad \hat{\lambda}_2=1+[(6-n)/(3n+22)]\epsilon_3/2, \tag{9}$$

in agreement with the general formulas for $n=1$ given in (5).

(iv) *Odd-dominated Ising systems.*—In addition to the usual *even* fixed-point Hamiltonians described above, (1) admits (for $n=1$) fixed points which have leading *odd* terms. We may do $\epsilon_{\Theta-1/2}$ expansions for $\Theta=2,3,\ldots$ in this case as well. The fixed-point Hamiltonian is of order $(\epsilon_{\Theta-1/2})^{1/2}$. We write the fixed-point Hamiltonian H^* as

$$H^*=(\epsilon_{\Theta-1/2})^{1/2}v_\Theta h_{2\Theta-1}+\epsilon_{\Theta-1/2}v_\Theta^2 f_e+(\epsilon_{\Theta-1/2})^{3/2}v_\Theta^3 f_0+\ldots, \tag{10}$$

where $h_{2\Theta-1}$ is an *odd* Hermite polynomial, and f_e is an even and f_0 an odd function of s. Solving (1) to first order in $\epsilon_{\Theta-1/2}$, we find the fixed-point value v_Θ and the perturbed eigenvalues to be given by

$$1=-\tfrac{1}{6}dv_\Theta^2\langle\mathcal{G}(2\Theta-1,2\Theta-1)|2\Theta-1\rangle, \tag{11a}$$

$$\hat{\lambda}_{l/2}-\lambda_{l/2}=-3\epsilon_{\Theta-1/2}\frac{\langle\mathcal{G}(2\Theta-1,l)|l\rangle}{\langle\mathcal{G}(2\Theta-1,2\Theta-1)|2\Theta-1\rangle},\quad l=1,2,3,\ldots. \tag{11b}$$

The operator $\mathcal{G}$ in (11) is

$$\mathcal{G}(m,l)\equiv 64l(l-1)m(m-1)[(h_{m-2})^2h_{l-2}+h_{l-2}\mathcal{L}_m(h_{m-2})^2+2h_{m-2}\mathcal{L}_l(h_{l-2}\cdot h_{m-2})], \tag{12}$$

where $\mathcal{L}_l$ is defined by $\mathcal{L}_l h_p=[p(p-1)/p-l]h_{p-2}$ for all Hermite polynomials h_p. At least for $2\Theta-1=3,5$, we have $v_\Theta^2<0$; the Hamiltonian is real only if $\epsilon_{\Theta-1/2}<0$. For $\epsilon_{\Theta-1/2}>0$ the odd parts of the fixed-point Hamiltonian are purely imaginary.[17]

The Wegner-Houghton approximate renormalization group proposed here provides a straightforward framework in which to explore the consequences of the full renormalization group. As a differential representation, it is suited to investigations of nonlinear phenomena such as crossover competition between two or more fixed points. Elsewhere[18] we have solved (8) near $d=4$ for the *nonlinear* crossover between critical and Gaussian (mean-field) behavior. The extension to crossover from tricritical to mean-field behavior seems to be more difficult.

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[11]To see this, it is sufficient to note that the Q_j are eigenfunctions of the full linear renormalization-group operator. The powers of x in the Q_j are replaced by more complicated sums over momentum: $(nx)^p$ becomes

$$\sum_{k_1 k_1'} \cdots \sum_{k_p k_p'} (\vec{s}_{k_1} \cdot \vec{s}_{k_1'}) \cdots (\vec{s}_{k_p} \cdot \vec{s}_{k_p'}) \delta_{k_1+k_1'+\cdots k_p+k_p', 0}.$$

With these emendations, an examination of the full nonlinear renormalization-group equation of Ref. 7 shows that the fixed point and eigenvalues are correct to first order in $\epsilon_{\mathcal{O}}$, and $\eta_{\mathcal{O}}$ is $o(\epsilon_{\mathcal{O}}^2)$.

[12]Points (ii) and (iii) hold for general n; (i) cannot hold for *arbitrary* n {e.g., for $\mathcal{O}=2$, $\lambda_1 = 2 - [(n+2)/(n+8)]\epsilon_2$}.

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