Integral Formalism for Gauge Fields

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(Received 10 June 1974)

A new integral formalism for gauge fields is described. Further developments are presented, including gravitation equations related to, but not identical with, Einstein's equations.

It was pointed out by Weyl many years ago that the electromagnetic field can be formulated in terms of an Abelian gauge transformation. This idea was extended[1] in 1954 to the concept of gauge fields for non-Abelian groups. That formulation, like the Weyl formulation for electromagnetism, was based on the replacement of ∂_μ by $\partial_\mu - ieB_\mu$. One might call such formulations differential formulations. It is the purpose of the present paper to reformulate the concept of gauge fields in an *integral formalism.* The new formalism is conceptually superior to the differential formalism and allows for natural developments of additional concepts. It further allows a mathematical and physical discussion of the gravitational field *as a gauge field,* resulting in equations related, but not identical, to Einstein's.

The basic point is the fact that *electromagnetism is a nonintegrable phase factor,* a fact discussed many years ago by Dirac, Peierls, and others, and more recently by many authors.[2] This fact is now generalized as follows:

Definition of a gauge field.—Consider a manifold with points on it labeled by x^μ ($\mu = 1, 2, \ldots, n$) and consider a gauge G which is a Lie group with generators X_k ($k = 1, 2, \ldots, m$). [For $G = U(1)$ we have electromagnetism; for G non-Abelian we have non-Abelian gauge fields.] Define a path-dependent (i.e., nonintegrable) phase factor φ_{AB} as an element of the group G associated with path AB between two points A and B on the manifold. The association is to have the group property: $\varphi_{ABC} = \varphi_{AB}\varphi_{BC}$, where the paths AB and BC are segments of ABC. Furthermore for an infinitiesimal path A to $A + dx^\mu$ the phase factor is close to the identity I of G, so that[3]

$$\varphi_{A(A+dx)} = I + b_\mu{}^k(x) X_k \, dx^\mu . \tag{1}$$

The function $b_\mu{}^k(x)$ defined on the manifold will be called a *gauge potential;* φ_{AB} will be called a *gauge phase factor.*

With this definition additional concepts and theorems are naturally developed. We summarize some of these below. Details will be published elsewhere.

Gauge field strength.—Consider a path $ABCDA$ forming the border of an infinitesimal parallelogram with sides dx and dx'. φ_{ABCDA} can be computed by multiplying four phase factors like (1) together, resulting in

$$\varphi_{ABCDA} = I + f_{\mu\nu}{}^k X_k \, dx^\mu dx^{\nu\prime} , \tag{2}$$

where

$$f_{\mu\nu}{}^k = \frac{\partial b_\mu{}^k}{\partial x^\nu} - \frac{\partial b_\nu{}^k}{\partial x^\mu} - b_\mu{}^i b_\nu{}^j C_{ij}{}^k = -f_{\nu\mu}{}^k \tag{3}$$

in which $C_{ij}{}^k$ is the structure constant of G:

$$X_k X_j - X_j X_k = C_{kj}{}^i X_i . \tag{4}$$

$f_{\mu\nu}{}^k$ will be called a *gauge field,* or gauge field strength. They are the Faraday-Maxwell fields when $G = U(1)$.

Gauge transformation.—A gauge transformation in the integral formalism is defined by a transformation

$$\varphi_{AB} \to \varphi_{AB}' = \xi_A \varphi_{AB} \xi_B{}^{-1} , \tag{5}$$

where ξ_A is an element of G which depends on the point A. It is clear that under (5)

$$\varphi_{ABCDA} \to \varphi_{ABCDA}' = \xi_A \varphi_{ABCDA} \xi_A{}^{-1} . \tag{6}$$

Thus

$$f_{\mu\nu}{}^{k\prime} = \langle k | R_{\mathrm{adj}} | j \rangle f_{\mu\nu}{}^j , \tag{7}$$

where R_{adj} is the adjoint representation for the element ξ_A. The simple transformation property (7) is the definition for the concept that $f_{\mu\nu}{}^k$ is *gauge covariant.* Generalization to other representations R of G for a gauge-covariant quantity $\psi_{\alpha\beta\gamma}{}^K$ is immediate[3]:

$$\psi_{\alpha\beta\gamma}{}^{K\prime} = \langle K | R(\xi_A) | J \rangle \psi_{\alpha\beta\gamma}{}^J . \tag{8}$$

$b_\mu{}^k$ is not gauge covariant; $f_{\mu\nu}{}^k$ is.

Gauge-covariant differentiation.—To *retain* *gauge covariance* in differentiation we define

$$\psi^K|_\mu = \frac{\partial \psi^K}{\partial x^\mu} + b_\mu{}^k \langle K|Z_k|J\rangle \psi^J, \tag{9}$$

where Z_k is the matrix representation of X_k. Generalization to other cases is obvious. An interesting theorem is that

$$f_{\mu\nu}|_\lambda{}^k + f_{\nu\lambda}|_\mu{}^k + f_{\lambda\mu}|_\nu{}^k = 0, \tag{10}$$

which is the gauge-Bianchi identity.

Introduction of a Riemannian metric.—So far we need no metric for the manifold. Now we introduce a metric for it and discuss arbitrary coordinate transformations. We come then naturally to *Riemannian covariant* quantities and *doubly covariant derivatives.* $b_\mu{}^k$ is Riemannian covariant, since φ_{AB} is coordinate-system independent. $f_{\mu\nu}{}^k$ is doubly covariant. We have

$$\begin{aligned} \psi^K{}_{\Vert\mu} &= \psi^K|_\mu, \\ \psi^{K\nu}{}_{\Vert\mu} &= \psi^{K\nu}|_\mu + \left\{ \begin{matrix} \nu \\ \mu\alpha \end{matrix} \right\} \psi^{K\alpha}, \\ f_{\mu\nu\Vert\lambda}{}^k &= f_{\mu\nu}|_\lambda{}^k - \left\{ \begin{matrix} \alpha \\ \mu\lambda \end{matrix} \right\} f_{\alpha\nu}{}^k - \left\{ \begin{matrix} \alpha \\ \nu\lambda \end{matrix} \right\} f_{\mu\alpha}{}^k, \end{aligned} \tag{11}$$

etc. It is easily shown that

$$f_{\mu\nu\Vert\lambda}{}^k + f_{\nu\lambda\Vert\mu}{}^k + f_{\lambda\mu\Vert\nu}{}^k = 0 \tag{12}$$

which is satisfied by *all* gauge fields on *all* Riemannian manifolds.

Source of gauge fields.— We *define,* in analogy with electromagnetism, a source four-vector $J_\mu{}^k$ for a gauge field:

$$J_\mu{}^k = g^{\nu\lambda} f_{\mu\nu\Vert\lambda}{}^k = f_{\mu\nu}{}^{k\Vert\nu}. \tag{13}$$

After some computation one derives a theorem:

$$g^{\mu\lambda} J_{\mu\Vert\lambda}{}^k = 0 \quad \text{(conserved current)}, \tag{14}$$

which in electromagnetism states charge conservation. In Ref. 1 this was Eq. (14). One can also generalize Eqs. (15) and (16) of Ref. 1, leading to the concept of "total charge."

Parallel-displacement gauge field.—For any Riemannian manifold, the important concept of parallel displacement defines, along any path AB, a *linear* relationship between any vector V_A at A and its parallel vector V_B at B. Thus parallel displacement is defined by an $n\times n$ matrix M_{AB} which gives this linear relationship. M_{AB} is a representation of an element of GL(n). Thus we have the following:

Theorem.—Parallel displacement defines a gauge field with G being GL(n). The index k has n^2 values and we write $k=(\alpha\beta)$. The gauge potential and gauge fields are respectively

$$b_\mu{}^{(\alpha\beta)} = \left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\}, \quad f^{(\alpha\beta)}{}_{\mu\nu} = -R^\alpha{}_{\beta\mu\nu}. \tag{15}$$

It is important to recognize that in this definition we have chosen a fixed coordinate system. A coordinate transformation would generate a linear transformation in the vector spaces V_A and V_B. In other words $M_{AB} \to N_A M_{AB} N_B{}^{-1}$. Comparison with (5) shows thus that a coordinate transformation generates a simultaneous gauge transformation of the parallel-displacement gauge potential. In fact, the usual nonlinear term in the transformation of $\{{}^{\alpha}_{\alpha\gamma}\}$ is precisely the nonlinear term needed in the gauge transformation of the gauge noncovariant quantity $b_\mu{}^{(\alpha\beta)}$. In this connection we observe that for GL(n),

$$C^{(\alpha\beta)}_{(\lambda\mu)(\eta\zeta)} = \delta_{\mu\eta}\delta_{\alpha\lambda}\delta_{\beta\zeta} - \delta_{\lambda\zeta}\delta_{\alpha\eta}\delta_{\beta\mu}. \tag{16}$$

Thus by definitions (9) and (11)

$$\begin{aligned} \psi^{(\alpha\beta)}{}_{\Vert\mu} &= \frac{\partial \psi^{(\alpha\beta)}}{\partial x^\mu} + b_\mu{}^{(\lambda\nu)} C^{(\alpha\beta)}_{(\lambda\nu)(\eta\zeta)} \psi^{(\eta\zeta)} \\ &= \psi^\alpha{}_{\beta;\mu}, \end{aligned} \tag{17}$$

where the semicolon represents the usual Riemannian covariant differentiation with α and β treated as usual contravariant and covariant indices. The rule works also in general. E.g.,

$$f^{(\alpha\beta)}{}_{\mu\nu\Vert\lambda} = -R^\alpha{}_{\beta\mu\nu;\lambda}. \tag{18}$$

Nontrivial sourceless gauge fields.—Gauge fields for which $f_{\mu\nu}{}^k \neq 0$ and $J_\mu{}^k = 0$ are of physical interest. So far only nonanalytic examples are known.[4]

We now can construct two general types of general types of examples.

(a) Consider the natural Riemannian geometry of a semisimple Lie group. Its parallel-displacement gauge field is sourceless and analytic.

(b) Consider the same Riemannian manifold of a group G as above in (a). Define φ_{AB} as that for an infinitesimal path AB, $\varphi_{AB} = (A^{-1}B)^{1/2}$. This gauge phase factor which is itself an element of G gives a gauge field which is analytic and sourceless.

Pure spaces.—A Riemannian manifold for which the parallel-displacement gauge field is sourceless will be called a pure space. A necessary and sufficient condition for a pure space is

$$R_{\mu\alpha;\beta} = R_{\mu\beta;\alpha} \tag{19}$$

A four-dimensional Einstein space, i.e., one for which $R_{\alpha\beta}=0$, is a pure space.

Gravitational field as a gauge field.—The electromagnetic field and the usual gauge fields are special cases of gauge fields, satisfying (12) and (13). A natural question is whether one should identify these *same equations* for the parallel-displacement gauge field as the equations for the gravitational field. There are advantages in this identification and we shall come back to this topic in a later communication. If one adopts this identification then gravitational equations are third-order differential equations[5] for $g_{\mu\nu}$. A pure gravitational field is then described by a pure space as defined above.

Variational principles.—Equation (13) with $J_\mu^k=0$ follows from a variational principle $\delta\int\sqrt{-g}\,d^nx=0$, where

$$L=f_{\mu\nu}{}^k f_{\alpha\beta}{}^j g^{\mu\alpha}g^{\nu\beta}C_{ka}{}^bC_{jb}{}^a. \tag{20}$$

In the variation $g_{\mu\nu}$ is kept fixed and $b_\mu{}^k$ is varied, and $f_{\mu\nu}{}^k$ is given by (3); $C_{ka}{}^b$ are not varied. One could also find a variational principle which is satisfied by a pure space (19). Choose $C_{ka}{}^b$ to be the structure constants for GL(n), given by (16). Write the L of (20) as a functional of $b_\mu{}^{(\alpha\beta)}$ and $g^{\lambda\nu}$:

$$L=L(b_\mu{}^{(\alpha\beta)},g^{\lambda\nu}), \tag{21}$$

which of course also contains derivatives of $b_\mu{}^k$ and $g^{\lambda\nu}$. Now form the variation

$$\delta\int\left[L(b_\mu{}^{(\alpha\beta)},g^{\lambda\nu})-L\left(\left\{\begin{matrix}\alpha\\ \beta\mu\end{matrix}\right\},g^{\lambda\nu}\right)\right]\times\sqrt{-g}\,d^nx=0, \tag{22}$$

in which $b_\mu{}^{(\alpha\beta)}$ and $g^{\lambda\nu}$ are independently varied. The resultant equations are satisfied by (15) and (19).

It is a great pleasure to acknowledge the warm hospitality extended to me during my visit to the Institute of Theoretical Physics at Wrocław where this paper was written.

[1] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2] S. Mandelstam, Ann. Phys. (New York) 19 1, 25 (1962); I. Białynicki-Birula, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. 11, 135 (1963).

[3] We use the summation convention for repeated indices. Greek indices run from 1 to n. Lower case Latin indices run from 1 to m. m of course is also the dimension of the adjoint representation of G. Upper case Latin indices run from 1 to M, where M is the dimension of a representation of G.

[4] T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions,* edited by H. Mark and S. Fernbach (Wiley, New York, 1969), p. 349.

[5] R. Utiyama, Phys. Rev. 101, 1957 (1956), had concluded that Einstein's equations are gauge-field equations. We believe that was an unnatural interpretation of gauge fields.