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Size of Barely Bound Many-Body Systems

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(Received 23 May 1974)

It is pointed out that, in contrast to the case of S-wave two-body bound states, the size of a many-body bound state remains finite (irrespective of its angular momentum) even if its binding energy vanishes. Physical implications for barely bound many-body systems are outlined.

The size of an S-wave two-body bound state diverges as the binding energy of the bound state vanishes.[1] This has an important physical implication, recognized long ago in connection with the first investigations of the deuteron[2]: The size of an S-wave barely bound state is approximately $1/(2\mu|E|)^{1/2}=1/q$, where $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass of the two-body system and E is its binding energy ($\hbar=c=1$). Thus any (reasonable) model of the neutron-proton interaction that fits the deuteron binding energy yields a deuteron wave function having essentially the same extension, $\sim 1/q$ (much larger than the range r_0 of the interaction[3]). Therefore any additional information on the neutron-proton interaction, besides that implied by the value of the binding energy, can result only from measurements of the deuteron size that are sufficiently accurate to display corrections of order qr_0 relative to the dominant term.[2,4]

This phenomenon is peculiar to S-wave two-body bound states. In fact, the wave function of a two-body l-wave bound state is proportional, for $r \gg r_0$ (where r is the interparticle separation, and r_0 is the range of the forces[5]), to the (asymptotically vanishing) free solution of the l-wave Schrödinger equation, $(qr)^{-1/2}K_{l+1/2}(qr)$,[6] and therefore it becomes $c(2l-1)!!(qr)^{-(l+1)}$ for $r_0 \ll r \ll 1/q$ and $c(qr)^{-1}\exp(-qr)$ for $r \gg 1/q$, c being a constant whose value is determined by the behavior of the wave function for $r \lesssim r_0$, and by the normalization condition. It follows that, in the zero-energy case, the wave function is asymptotically proportional to $r^{-(l+1)}$, being therefore, for $l>\frac{1}{2}$, still normalizable, and implying that the expectation value of $r^{|p|}$ is finite for $|p|<p_0=2l-1$. Moreover, simple power counting shows that, for an l-wave two-body barely bound state (i.e., such that $qr_0 \ll 1$), the expectation value of $r^{|p|}$ is of order $r_0^{\,|p|}$ for $|p|<p_0$, $-r_0^{\,|p|}\ln(qr_0)$ for $|p|=p_0$, and $r_0^{\,|p|}(qr_0)^{p_0-|p|}$ for $|p|>p_0$. These estimates refer to the case $l>\frac{1}{2}$, i.e., when the normalization integral remains finite for $q=0$; note that in all cases the result depends on r_0. For $l<\frac{1}{2}$ one finds instead, as a result of the divergence of the normalization integral for $q=0$, that the expectation value of $r^{|p|}$, is of order $q^{-|p|}$, i.e., independent of the value of r_0 (provided $qr_0 \ll 1$).

These results display the exceptional nature of the two-body S-wave case, and imply that even rough measurements on two-body higher-wave barely bound states would yield more information on the forces than that conveyed by the binding-energy value—in contrast to the S-wave case.

This property of higher-wave barely bound states originates from the normalizability of the bound-state wave function in the zero-energy case, or, equivalently, from the finite size of the zero-energy bound state. The purpose of this Letter is to point out that, in analogy to the two-body higher-wave case, and in contrast to the two-body S-wave case, *the size of an N-body ($N > 2$) bound state*[7] *remains finite (irrespective of its angular momentum) when its binding energy vanishes, provided no $(N-n)$-body* $(0 < n < N-1)$ *bound state exists.*[8,9] We prove here this result for $N=3$, and outline the proof for $N>3$. For simplicity we prove the result for spinless identical particles interacting via two-body potentials, but none of these restrictions is essential for its validity.[10]

The physical implications of this result refer to barely bound many-body bound states, and are analogous to those indicated above in the higher-wave two-body case. They are outlined in the last part of this Letter; we hope to be able in the near future to publish a more complete treatment.

The proof for the three-body case is based on the homogeneous Faddeev equation[11]:

$$\chi(\vec{k},\vec{p},E) = -m\int d^3p'[t(\vec{k},-\tfrac{1}{2}\vec{p}-\vec{p}',E-\tfrac{3}{4}p^2/m)\chi(\vec{p}+\tfrac{1}{2}\vec{p}',\vec{p}',E) + t(\vec{k},\tfrac{1}{2}\vec{p}+\vec{p}',E-\tfrac{3}{4}p^2/m)\chi(-\vec{p}-\tfrac{1}{2}\vec{p}',\vec{p}',E)]/(p^2+p'^2+\vec{p}\cdot\vec{p}'-mE). \quad (1)$$

Here $\vec{k}$ and $\vec{p}$ are connected to the momenta of the three particles in the usual way, $\vec{k}=\frac{1}{2}(\vec{k}_2-\vec{k}_3)$, $\vec{p}=\frac{1}{3}(\vec{k}_2+\vec{k}_3-2\vec{k}_1)$, E is the total energy in the c.m. frame, and $\chi(\vec{k},\vec{p},E)$ is related to the Fourier-transformed three-body wave function $\Psi(k_1,k_2,k_3)$ by the relation

$$\Psi(\vec{k}_1,\vec{k}_2,\vec{k}_3) = [(1+P_{12}+P_{23})\chi(\vec{k},\vec{p},E)]/(\tfrac{3}{4}p^2+k^2-mE), \quad (2)$$

where P_{ij} is the operator that permutes particles i and j, and m is the mass of the particles. Thus the normalization condition for the bound state sets the integral

$$I=\int d^3k\,d^3p\,|\chi(\vec{k},\vec{p},E)+\chi(-\tfrac{1}{2}\vec{k}-\tfrac{3}{4}\vec{p},-\tfrac{1}{2}\vec{p}+\vec{k},E)+\chi(-\tfrac{1}{2}\vec{k}+\tfrac{3}{4}\vec{p},-\tfrac{1}{2}\vec{p}-\vec{k},E)|^2/(\tfrac{3}{4}p^2+k^2-mE)^2 \quad (3)$$

equal to a constant. The task is to prove that I remains convergent (at $k\simeq 0$, $p\simeq 0$) even for a solution of Eq. (1) with $E=0$.

The function $t(\vec{k},\vec{p},E')$ in Eq. (1) is the two-body off-shell t matrix, and it is certainly finite at $k=0$, $p=0$. It is moreover nonsingular for $E'<0$, and finite for $E'=0$, *provided there are no two-body bound states.* But then Eq. (1) implies that $\chi(\vec{k},\vec{p},E)$ is also finite at $k=p=E=0$, and then simple power counting shows that I remains convergent at $k\approx 0$, $p\approx 0$ even for $E=0$, Q.E.D.

Using the fact that $\chi(\vec{k},\vec{p},0)$ is finite at $k=p=0$, one can easily evaluate the asymptotic behavior of the three-body wave function $\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)$:

$$\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)\sim \text{const}(r_{12}^2+r_{23}^2+r_{31}^2)^{-2}, \quad (4)$$

implying again convergence at large r_{ij} of the normalization integral

$$\int d^3r_1\,d^3r_2\,d^3r_3\,\delta(\vec{r}_1+\vec{r}_2+\vec{r}_3)|\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)|^2. \quad (5)$$

Note that Eq. (4) corresponds simply to the behavior at large r_{ij} of the (asymptotically vanishing) free solution of the zero-energy three-body Schrödinger equation; indeed for negative energy E and in the N-body case, this solution is just the free translation-invariant Green's function[12]

$$G_N(E;\vec{r}_1,\ldots,\vec{r}_N) = -2m(2\pi)^{-3(N-1)/2}q^{(3N-5)/2}\rho^{-(3N-5/2)}K_{(3N-5)/2}(q\rho), \quad (6)$$

where m is the mass of the particles, $q^2=2m|E|$, and

$$\rho^2=\sum_{i>j=1}^{N}|\vec{r}_i-\vec{r}_j|^2/N.$$

This remark displays the connection of the many-body case with the higher-wave two-body case discussed above, indicating the similarity of the mechanisms that bring about the convergence of the zero-energy normalization integrals.[13] This same remark indicates that the result must be true for $N>3$; a formal proof can be based on the Faddeev-Jakubowski equations,[14] the requirement of nonexistence of $(N-n)$-body bound states being related to the occurrence of $(N-n)$-body off-shell t matrices as kernels in these equations.

The Faddeev (and Faddeev-Jakubowski) equations provide the most convenient tool to prove our result, since they neatly display the role played by the requirement that there be no $(N-n)$-body bound states. It is, however, illuminating to also try to understand in more physical, if less rigorous, terms the mechanism whereby the presence of $(N-n)$-body bound states alters the asymptotic behavior of the zero-energy many-body wave function.

Let us refer, for simplicity, to the three-body case, and focus our attention on the physically more interesting case of a barely bound state (i.e., such that $qr_0 \ll 1$, with $q^2 = 2m|E|$). Then if a two-body bound state, with (negative) energy $E_2 > E$, exists, the three-body bound-state wave function contains asymptotically several terms (that give the dominant behaviors in different sectors of configuration space),[15] one of which is proportional to $(q_1\xi)^{-1/2}K_{l+1/2}(q_1\xi)\varphi(\eta)$, with, say, $\vec{\xi} = 6^{-1/2}(\vec{r}_2 + \vec{r}_3 - 2\vec{r}_1)$, $\vec{\eta} = 2^{-1/2}(\vec{r}_2 - \vec{r}_3)$, $q_2^2 = m|E_2|$, and $q_1^2 + 2q_2^2 = q^2$. This term is of course to be interpreted as a quasi-two-body bound state of angular momentum l, composed of particle 1 and the two-body bound state (with wave function φ) made up of particles 2 and 3. The size associated to this term will then be much larger than r_0, if the bound state of particles 2 and 3 has zero angular momentum (in which case it would have a size of order $1/q_2 > 1/q \gg r_0$), or if $l = 0$ (in which case it would have a size of order $1/q_1 > 1/q \gg r_0$), or if both things happen (in which case it would have a size or order $1/q_1$ or $1/q_2$, whichever is larger).[16] It is also clear, on the grounds of physical continuity, that similar estimates of the size of the three-body barely bound state obtained under the hypothesis that, in place of two-body barely-bound states, there exist two-body, very low-energy resonances or virtual states.[17] But if instead no two-body barely bound states, nor virtual states, nor very low-energy resonances exist, then our result implies that the size of a three-body barely bound state is of order $r_0 \ll 1/q$.[18]

As indicated by these considerations, the main phenomenological implication of our remark relates to the size of barely bound N-body systems; it would therefore lead to specific predictions for several observables involving such systems (form factors, vertex functions, transition probabilities, reactions). Suffice it here to note that the main parameters affecting such predictions are the expectation values of $r^{|p|}$, and that for these quantities, under the conditions implied by the above discussion, exactly the same results as given above for a two-body l-wave barely bound state hold, except for a redefinition of p_0, that now reads $p_0 = 2K_{\min} + 3N - 7$.[19] Here $K_{\min}$ is a nonnegative integer that is uniquely determined by the angular momentum, parity, etc. of the N-body state, and by the properties of the particles that constitute it (in particular, the kind of statistics they satisfy); it equals the minimal value taken by the sum of the quantum numbers l_j characterizing the angular momenta of the Jacobi coordinates.[20]

Finally we mention that, while in this paper we have focused on bound states, many of the phenomenological considerations also apply to (very low-energy) N-body resonances. A convenient technique to display the analogy is the Hilbert-Schmidt technique, that allows one to associate normalizable wave functions also to resonances, and therefore provides a firm ground for the discussion of properties related to their size.[21]

It is a pleasure to acknowledge useful conversations on the subject of this paper with A. M. Badalyan. One of us (F.C.) would like to thank the Institute of Theoretical and Experimental Physics for the kind hospitality provided during a one-month visit, effected in the framework of the Comitato Nazionale per l'Energia Nucleare–Goskomitet po Islol'zovaniyu Atomnaya Energii scientific exchange agreement.

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[1]Throughout this paper, for simplicity of language, we refer to zero-energy two-body S-wave states, even though, as discussed below, the corresponding wave functions are not normalizable (so that, strictly speaking, one should refer to such states as zero-energy resonances).

[2]E. P. Wigner, Z. Phys. 83, 253 (1933), and Phys. Rev. 43, 252 (1933); H. A. Bethe and R. Peierls, Proc. Roy. Soc., Ser. A 149, 176 (1935).

[3]For the deuteron, $1/q = 4.3$ fm; the pion Compton wavelength λ, that sets the scale of the *longer*-range part of the nucleon-nucleon interaction, is less than one third of that ($\lambda = 1.4$ fm).

[4]The Wigner-Bethe-Peierls argument applies only to the S-wave part of the deuteron wave function (see below); indeed any property of the deuteron related to the (small) D-wave part of the wave function does provide additional information on the nucleon-nucleon potential, besides that implicit in the value of the deuteron binding energy.

[5]We are assuming, for simplicity, that the potentials vanish exponentially at infinity; but for the validity of the main result it is sufficient that they be integrable at long range (i.e., in three-dimensional space, that $r^{3+\epsilon}V(r)$ vanish asymptotically for some $\epsilon > 0$). The exclusion of long-range potentials (for instance, Coulomb) is of course essential.

[6]Here, and in the following, $K_\nu(Z)$ is the modified Bessel function of the third kind. Also note that, where-

ever we mention the wave function, we understand the full (three-dimensional) wave function, namely, that normalized by $\int d^3r|\Psi|^2=1$.

[7]Here, and always in the following, we exclude the exceptional case of positive-energy bound states.

[8]This result (for $N=3$) is implicitly contained in a report given some years ago by one of us [Yu. A. Simonov, in *Proceedings of the Problem Symposium of Nuclear Physics, Tbilisi, U.S.S.R., 1967,* Vol. I, p.7], and has been explicitly evinced from this reference by V. N. Efimov, who (in an unpublished seminar) metioned this fact as the reason why the Efimov effect {V. N. Efimov, [JETP Lett. 16, 34 (1972)], and Nucl. Phys. A210, 157 (1973)} is not expected to occur for $N>3$. It should be emphasized that the result reported here for zero-energy bound states cannot be obtained by a simple limiting process from the results given in the recent papers dealing with the asymptotic behavior of the three-body wave function, that consider only nonzero-energy states: S. P. Merkuriev, Teor. Mat. Phys. 8, 235 (1971), and Yad. Fiz. 19, 447 (1974) [Sov. J. Nucl. Phys., to be published]; R. G. Newton, Ann. Phys. (New York) 74, 324 (1972).

[9]The condition excluding $(N-n)$-body bound states might appear trivial, as a result of the assumed existence of a zero-energy N-body bound state (see also Ref. 7). But the critical assumption for the validity of the theorem consists in the exclusion of *zero-energy* $(N-n)$-body bound states; and this is not an altogether exceptional eventuality, since it is for instance known that a zero-energy two-body bound state implies the existence of an infinite number of three-body bound states, whose energies accumulate at zero (this is the Efimov effect; see Ref. 8). The importance of this condition relates however mainly to the physically more relevant case of an N-body barely bound state, i.e., a state with nonvanishing, but very small, binding energy (see below). Let us also emphasize that the N-body bound state under consideration need not be the ground state of the N-body system.

[10]We treat for simplicity only the case of ordinary (three-dimensional) space; the structure of the proof implies that the main result remains valid in d-dimensional space, provided $d(N-1)>4$.

[11]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)].

[12]See, for instance, F. Calegero and Yu. A. Simonov, Nuovo Cimento 56B, 71 (1968). We refer here, for simplicity, to the case of spinless bosons. For the more general case, one must introduce the quantum number $K_{\min}$ (see below).

[13]The connection between the many-body bound-state problem and the higher-wave two-body problem has been been already pointed out and exploited in a different, but related, context: F. Calogero and Yu. A. Simonov, Phys. Rev. 169, 789 (1968).

[14]O. A. Jakubowsky, Yad. Fiz. 5, 1312 (1967) [Sov. J. Nucl. Phys. 5, 937 (1967)]; L. D. Faddeev, in *Three-Body Problem in Nuclear and Particle Physics,* edited by J. S. C. McKee and P. M. Rolph (North-Holland, Amsterdam, 1970), p. 154.

[15]See, for instance, Merkuriev, Ref. 8, where, however, the function $z^{-1/2}K_{l+1/2}(z)$ is replaced by its asymptotic part $z^{-1}\exp(-z)$.

[16]An example of the last kind (and with $q_1 \ll q_2$) is hypertritium ($E_{\rm deut}=-2.23$ MeV, $E_{\rm hyptrit}\approx -2.3$ MeV).

[17]One such example might be the trineutron, if it exists as a bound state (or resonance; see below).

[18]These arguments indicate that, if selection rules prevent a barely bound state for dissociating into S-state components, then its size would be of order $r_0 \ll 1/q$ even if two-body bound states or resonances exist. A three-body bound state with odd angular momentum and positive orbital parity is one such example.

[19]Of course the coefficient of the term giving the dominant contribution for small q depends on N, and on the precise definition of r.

[20]See, for instance, the report by Yu. A. Simonov, in *The Nuclear Many-Body Problem,* Proceedings of the Symposium on Present Status and Novel Developments in the Nuclear Many-Body Problem, Rome, 1972, edited by F. Calogero and C. Ciofi degli Atti (Editrice Compositori, Bologna, 1974), Vol. 1, p. 527. For instance, for a 1^- system composed of three pions (like the ω meson), $K_{\min}=2$; for a 0^- state of four identical fermions, $K_{\min}=3$; and of course for the two-body case, $K_{\min}=l$.

[21]See, for instance, S. Weinberg, Phys. Rev. 131, 440 (1936), and, for a more detailed discussion of three-body resonances, A. M. Badalyan and Yu. A. Simonov, Yad. Fiz. 17, 441 (1973), and 18, 73 (1973) [Sov. J. Nucl. Phys. 17, 225 (1973), and 18, (1973)].