Contribution to Sideband Instability Theory

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The O'Neil trapping regime is known to take place only after a finite damping of the main wave. As a consequence of that finite damping, one obtains a beam-type distribution function for the detrapped particles. By including in the stability analysis this detailed distribution function, we give a new interpretation of the sideband generation.

The original observations by Wharton, Malmberg, and O'Neil[1] that satellite frequencies of a large-amplitude collisionless electron plasma wave are unstable were interpreted in terms of trapped-electron instability.[2,3] From that time, numerical simulations[4] and laboratory experiments[5-7] have made more precise the features of this instability. This instability is very sensitive to the time behavior of the main wave. It has been shown[8] that several regimes are possible according to the values of $|\gamma_L/\omega_{b0}|$, where γ_L is the Landau damping rate and ω_{b0} the initial electron bounce frequency: $\omega_{b_0}=k_0\{|e|\Phi_0/m_e\}^{1/2}$. Firstly, if $|\gamma_L/\omega_{b0}|>1$, the main wave is Landau damped. Secondly, if $|\gamma_L/\omega_{b_0}|\ll 1$, the main-wave amplitude oscillates around a constant mean value (O'Neil's regime[9]). Lastly, if $0.2 \lesssim |\gamma_L/\omega_{b_0}| \lesssim 0.7$, the potential-wave amplitude decreases first and oscillates after a while, whereas one or three sidebands grow with frequencies approximately given by $\omega - kV_{\Phi_0} \simeq \pm\omega_{b_0}, 2\omega_{b_0}$. Here, V_{Φ_0} is the main-wave phase velocity. Measurements of the sideband growth rate[1,5,6] show that, depending on the initial-wave potential, one obtains $\gamma_S \propto \omega_{b_0}{}^n$ with $1 \leqslant n \leqslant 2.4$. Moreover, after the initial main-wave damping, the electron distribution function exhibits an accumulation of untrapped particles at the edge of the well.[4]

These experimental features suggest the following mechanism to explain the sideband instability. For simplicity, we restrict ourselves to the case $\mu = V_{\Phi_0}\omega_{b0}/k_0 V_{Te}{}^2 < 1$ (V_{Te} is the electron thermal velocity), which enables us to neglect the nonlinear frequency shift.[10] Let us assume an initially Maxwellian plasma where a monochromatic wave is injected at $t=0$. During the first-half bounce period, trapped electrons with velocities $V<V_{\Phi_0}$ are accelerated by the wave electric field, whereas electrons with $V>V_{\Phi_0}$ are decelerated. If the damping rate is small compared to the bounce time, the net energy exchange between wave and particles tends to be null. Conversely, if these two scales are comparable, many of the accelerated particles are not reflected during the second-half bounce period, and they become a beam of detrapped particles. On the other hand, reflected particles lose kinetic energy to the wave, inducing a new rise of the main-wave amplitude. A lower-amplitude oscillatory regime takes place when the slope of the distribution function of the remaining trapped particles is small enough. Then the main effect of the wave damping is to give rise to a beam of detrapped particles, which we show to be responsible for the sideband instability.

Before proceeding with the stability analysis, we calculate the distribution function at the end of the damping stage. In order to be able to solve the particle equation of motion in the wave frame,

$$\frac{d^2x}{dt^2} = -\frac{|e|}{m_e}k_0\Phi(t)\sin(k_0x), \qquad (1)$$

we choose a particular time dependence for $\Phi(t)$:

$$\frac{1}{\Omega^2(t)}\frac{\partial\Omega(t)}{\partial t} = 2\epsilon = \text{const}, \qquad (2)$$

with

$$\Omega(t) = k_0[|e|\Phi(t)/m_e]^{1/2}.$$

Since in most experiments the oscillatory regime is reached after half a mean bounce period, we use Eq. (2) for $0<t\leqslant\overline{\tau}/2$, where $\overline{\tau}/2$ is defined by $\pi = \int_0^{\overline{\tau}/2}\Omega(t)\,dt$. For $t>\overline{\tau}/2$, we assume a constant amplitude, $\Phi_1 = \Phi(\overline{\tau}/2)$. We have to assume $|\epsilon| < 1$, otherwise the wave will be completely damped before one trapping oscillation could take place. On the other hand, $|\epsilon|$ must not be too small, otherwise the time variation of the potential is adiabatic, and only a negligible number of particles are detrapped. The parameter ϵ will be determined from the energy balance.

First we consider Eq. (1) for $t<\overline{\tau}/2$. Setting $y=\int_0^t\Omega(t)\,dt$, we have to solve

$$\frac{d^2k_0x}{dy^2} + 2\epsilon\frac{dk_0x}{dy} + \sin k_0x = 0. \qquad (3)$$

This is the equation of a pendulum with a negative friction term. It has been solved by Bogolioubov and Mitropolski.[11] Expanding $\sin k_0 x$, one shows that the solution which is obtained when retaining only the linear terms describes the motion of 95% of the initially trapped particles within an accuracy of better than 5%. Thus, in the following we use this approximation. If we set $\kappa^2 = 2|e|\Phi(t)/[W+|e|\Phi(t)]$, where W is the particle energy, we deduce, from Eq. (3), $\kappa_1^2 = \kappa^2(\overline{\tau}/2) = \kappa^2(0)\exp(2\epsilon\pi)$. For a stationary potential, one checks easily that $\kappa^2 < 1$ for untrapped particles and $\kappa^2 > 1$ for trapped particles. Here, because of the finite potential variation, we must distinguish three kinds of trajectories in the phase plane (x, V): (1) Particles with $\kappa_1^2 < e^{2\epsilon\pi}$ are always untrapped. For these particles we neglect the modulation of their velocity v in the wave frame. Thus their distribution function remains Maxwellian. (2) Particles with $\kappa_1^2 > 1$ remain trapped, and $v(\overline{\tau}/2) = -v(t=0)e^{\epsilon\pi}$. (3) Particles with $e^{2\epsilon\pi} < \kappa_1^2 < 1$ are trapped at $t=0$, and untrapped at $t=\overline{\tau}/2$. Depending on their initial energy, these particles can suffer one or no reflection. For initially trapped particles, those having $e^{2\epsilon\pi} < \kappa_1^2 < e^{\epsilon\pi}$ are not reflected, whereas one reflection occurs for those with $e^{\epsilon\pi} < \kappa_1^2 < 1$. Then $v(\overline{\tau}/2) = -v(t=0)e^{\epsilon\pi}$ if $1 > \kappa_1^2 > e^{\epsilon\pi}$, and $v(\overline{\tau}/2) = v(t=0)e^{\epsilon\pi}$ if $e^{\epsilon\pi} > \kappa_1^2 > e^{2\epsilon\pi}$. Consequently the main wave energy absorption during the first-half bounce period is due to particles with $\kappa_1^2 > e^{\epsilon\pi}$, the distribution function of which exhibits an inverted slope at $t = \overline{\tau}/2$, as shown on Fig. 1(a). From the energy balance equation we evaluate the amplitude characteristic scale of variation, $\epsilon = \gamma_L/4\omega_{b0}$.

The time development of the sideband instability can be studied when the oscillatory regime is established (i.e., for $t \geq \overline{\tau}/2$). We use the same methods as Bud'ko, Karpman, and Shkylar.[3] As in Ref. 3, we find that for $\mu < 1$, the main contribution to the growth rate comes from the ergodic terms of the distribution function. But as a result of the detrapping process we have previously described, we get the ergodic distribution function shown on Fig. 1(b). As for $1 > \kappa^2 > e^{\epsilon\pi}$, the sign of the κ-dependent term is opposite to the one obtained in Ref. 3, and the instability soon

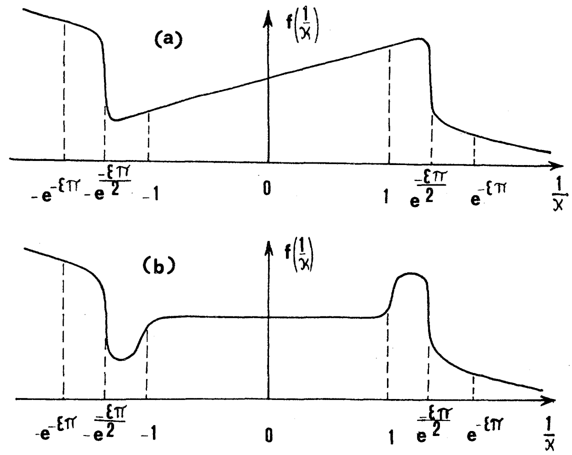

FIG. 1. Electron distribution in the wave frame as a function of κ^{-1} (κ having the sign of v). (a) At $t = \overline{\tau}/2$ the distribution exhibits an inverted slope compared to the initially Maxwellian function. (b) At $t = \infty$, the electrons exhibit a beam-type distribution function.

occurs even for $\mu \ll 1$. Nevertheless, provided that the contribution of ergodic terms dominates, it is convenient to work in the asymptotic time regime. As long as $\Phi_1/\Phi_0 = \exp(\pi\gamma_L/\omega_{b0})$ is not too small, the energy of detrapped particles is sharply peaked $[(\Phi_1/\Phi_0)^{1/4} < \kappa^2 < 1]$. Then their distribution function may be regarded as identical to that of a beam and the hole from which the beam came,

$$f_D = \pm \frac{\pi n_D \kappa^2 \sqrt{m_e}}{4K(\kappa)[|e|\Phi_1]^{1/2}} \delta(\kappa - \kappa_D), \tag{4}$$

where K is the complete elliptic integral and $2n_D$ is the number of detrapped particles,

$$\frac{n_D}{n_0} = \frac{2\omega_{b1}^2 \omega_{b0}}{\omega_p^2 \omega_0} \left(\frac{\gamma_L}{\omega_{b0}}\right)^2 \left(\frac{\Phi_0}{\Phi_1}\right)^{1/4}. \tag{5}$$

Here $\omega_p^2 = 4\pi n_0 e^2/m_e$, ω_0 is the main wave frequency, and κ_D is given by the balance equation which reads, with our approximation, $|\kappa_D|K(\kappa_D) = \frac{1}{2}\pi e^{\epsilon\pi/2}(-\epsilon\pi)^{-1/2}$.

Because of the spatial periodicity in the wave frame, all the Fourier components of the perturbed electric field $\tilde{E}(k+nk_0)$ are coupled together through the periodic modulation of the susceptibility.[12] In the laboratory frame the equation for $\tilde{E}(k, \omega)$ has the form

$$\sum_s [\mathcal{E}(k+sk_0, \omega+s\omega_0)\delta(l, s) + \chi_{l,s}]\tilde{E}(k+sk_0, \omega+s\omega_0) = 0, \tag{6}$$

where the fluid dispersion relation $\mathcal{E}(k, \omega) = 1 - \omega_p^2/(\omega^2 - 3k^2 V_T^2)$ describes the initially untrapped particles, for which we kept only zeroth order terms in Φ_1.[12] The ergodic trapped-particle distribution function is independent of κ to the lowest order in μ. Then the only contribution to the susceptibility $\chi_{l,s}$ comes from detrapped particles. We retain the most important components of the field, $\tilde{E}(k,\omega)$ and $\tilde{E}(k-2k_0,\omega-2\omega_0)$. Limiting ourselves to terms up to second order in $q_D=\exp[-\pi K'(\kappa_D)/K(\kappa_D)]$ the susceptibilities $\chi_{l,s}$ read

$$\chi_{0,0}=-\chi_{-2,-2}=-\frac{\pi^2E}{2K^3}\frac{\omega_D{}^2k_0U}{1-\kappa_D{}^2}(\omega-kV_{\Phi 0})A,\quad A=\frac{1-4q_D{}^2}{[(\omega-kV_{\Phi 0})^2-k_0{}^2U^2]^2}+\frac{16q_D{}^2}{[(\omega-kV_{\Phi 0})^2-4k_0{}^2U^2]^2}, \tag{7}$$

$$\chi_{0,-2}=-\chi_{-2,0}=-\frac{\pi^2E\omega_D{}^2k_0U(\omega-kV_{\Phi 0})}{2K^3(1-\kappa_D{}^2)}\frac{4q_D{}^2}{[(\omega-kV_{\Phi 0})^2-k_0{}^2U^2]^2}, \tag{8}$$

where ω_D is the beam plasma frequency, $U=\pi\omega_{b1}/|\kappa_D|K(\kappa_D)k_0$ is the averaged velocity of the beam, and K and E are complete elliptic integrals. Setting $\omega=\omega_0+\delta\omega$ and $k=k_0+\delta k$, the dispersion equation is solved to the first order in q_D. The maximum growth rate is obtained for $\delta k=\pm k_0U(V_{\Phi 0}/3V_T{}^2)$. In the laboratory frame

$$\gamma_{\Omega_b}=\frac{\sqrt{3}}{2}\omega_{b0}\left\{\frac{E\pi^2}{8K^3}\frac{e^{3\epsilon\pi}}{1-\kappa_D{}^2}\left(\frac{\gamma_L}{\omega_{b0}}\right)^2\frac{\omega_p{}^2}{\omega_0{}^2}\right\}^{1/3},$$
$$\omega-\omega_0=\pm\Omega_b=\pm\frac{k_0U}{1-V_{\Phi 0}{}^2/3V_T{}^2}\left\{1-\tfrac{1}{2}\left[\frac{E\kappa_D{}^3e^{-3\epsilon\pi}\omega_p{}^2}{8\pi(1-\kappa_D{}^2)\omega_0{}^2}\left(\frac{\gamma_L}{\omega_{b0}}\right)^2\right]^{1/3}\right\}. \tag{9}$$

Retaining now the second-order terms in q_D, we find that sidebands with frequencies $\omega-\omega_0\simeq\pm2\Omega_b$ are destabilized. Since $\gamma_{2\Omega_b}=(2q_D{}^2)^{1/3}\gamma_{\Omega_b}$, they will be observed only if κ_D is not too far from unity. Let us now take into account the sideband amplitude coupling terms. Solving the set of equations (6), $\tilde{E}(k,\omega)$ and $\tilde{E}(k-2k_0,\omega-2\omega_0)$ are coupled through the second-order terms in q_D:

$$\tilde{E}(\omega_0\pm\Omega_b,k)=\{-\chi_{0,-2}/[(\delta\mathcal{E})^2+\chi_{0,-2}{}^2]^{1/2}\mp\delta\mathcal{E}\}\tilde{E}(k-2k_0,\pm\Omega_b-\omega_0),$$

where

$$\delta\mathcal{E}=[(\delta\omega)^2-3(\delta k)^2V_{Te}{}^2]\omega_p{}^{-2}.$$

Then, in the laboratory frame, the red satellite has larger amplitude than the violet one, if $V_{\Phi_0}{}^2>3V_{Te}{}^2$, which is the usual experimental case. The ratio of $\tilde{E}(\omega_0-\Omega_b,k_0)$ to $\tilde{E}(\omega_0+\Omega_b,k_0)$ depends on q_D through $\chi_{0,-2}$. For very small values of q_D, $|\chi_{0,-2}|<|\delta\mathcal{E}|$ and $\tilde{E}(\omega_0+\Omega_b)\ll\tilde{E}(\omega_0-\Omega_b)$; while for $q_D<1$, but not too small, $|\chi_{0,-2}|>|\delta\mathcal{E}|$ and $\tilde{E}(\omega_0-\Omega_b)\sim\tilde{E}(\omega_0+\Omega_b)$.

Collecting the previous results, three sidebands should appear if γ_L/ω_{b0} is small ($\kappa_D{}^2\lesssim1$). Conversely, if γ_L/ω_{b0} is large ($\kappa_D{}^2\ll1$), only one sideband should appear. In this case the beam is wider, thus explaining the broadening of the spectra when only one sideband is observed.[5] This regime is rather limited in γ_L/ω_{b0}, at least if we maintain $\mu\ll1$. If $\mu\sim1$, nonlocal damping effects and nonlinear frequency shift will occur.[10] Then there should be one enhanced beam with $\kappa_D{}^2\ll1$, inducing a large-amplitude unique sideband wave. This regime is the natural extension of the regime $\mu<1$, $\kappa_D{}^2<1$, where only one sideband was found. Furthermore, when $\mu\sim1$, one can expect that the sideband growth rate will not be proportional to $\Phi_0{}^{1/2}$.[6]

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[1]C. B. Wharton, J. M. Malmberg, and T. M. O'Neil, Phys. Fluids 11, 1761 (1968).
[2]W. L. Kruer, J. M. Dawson, and R. N. Sudan, Phys. Rev. Lett. 23, 838 (1969).
[3]N. I. Bud'ko, V. I. Karpman, and D. R. Shkylar, Zh. Eksp. Teor. Fiz. 61, 1463 (1971) [Sov. Phys. JETP 34, 778 (1972)].
[4]W. L. Kruer and J. M. Dawson, Phys. Fluids 13, 2747 (1970).
[5]R. N. Franklin, S. M. Hamberger, H. Ikezi, G. Lampis, and G. J. Smith, Phys. Rev. Lett. 28, 1114 (1972).
[6]G. Jahns and G. Van Hoven, Phys. Rev. Lett. 31, 436 (1973).
[7]C. B. Wharton, J. M. Malmberg, and T. M. O'Neil, private communication.
[8]R. N. Franklin, S. M. Hamberger, and G. J. Smith, Phys. Rev. Lett. 29, 914 (1972).
[9]T. M. O'Neil, Phys. Fluids 8, 2255 (1965).
[10]G. J. Morales and T. M. O'Neil, Phys. Rev. Lett. 28, 417 (1972).
[11]N. Bogolioubov and I. Mitropolski, *Les Methodes Asymptotiques en Theorie des Oscillations Non Linéaire* (Gauthier-Villars, Paris, 1962).
[12]M. V. Goldman, Phys. Fluids 13, 1281 (1970).