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[14]The shape of the radial function $f(x_R)$ is sensitive to errors in the energy dependence of our normalization to the pp elastic cross section and in the energy calibration of our lead glass counter, especially near the phase-space boundary $x_R=1$. The absolute normalization of the data is reflected in the magnitude of the $p_\perp$ function.

Unified Description of Single-Particle Production in *pp* Collisions*

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We show that over a very wide range of c.m. angles, energies, and transverse momenta, the invariant cross sections for inclusive production of hadrons in pp collisions show a very simple behavior when expressed in terms of a new radial scaling variable $x_R \equiv p^*/p_{\max}^*$, where p^* is the total c.m. momentum of the hadron.

In a separate Letter,[1] we have presented data on inclusive π^0 production from pp collisions at the National Accelerator Laboratory (NAL), showing that for c.m. angles from 40 to 110°, transverse momenta from 0.3 to 4.3 GeV/c, and incident energies from 50 to 400 GeV, the invariant cross sections factorize very simply into the product of two universal functions:

$$E\, d^3\sigma/dp^3 = f(x_R)g(p_\perp), \tag{1}$$

where x_R is an angle-independent scaling variable,

$$x_R \equiv p^*/p_{\max}^* \approx 2p^*/\sqrt{s}. \tag{2}$$

The observed dependence on x_R can be approximated by

$$f(x_R) \propto (1-x_R)^4, \tag{3}$$

and the intrinsic transverse-momentum dependence by

$$g(p_\perp) \propto (p_\perp^2 + m^2)^{-4.5}, \quad m^2 = 0.86\ \mathrm{GeV}^2. \tag{4}$$

The simplicity of this factorization and the wide kinematic range over which we have found it applicable make it desirable to examine other pp inclusive data. Although the available data are limited, we feel it is important to provide an initial overview of an emerging pattern for inclusive measurements in strong interactions.

We have compared our results with other inclusive pion measurements near 90° in the c.m. system by the Chicago-Princeton group at NAL[2] ($\pi^{\pm}$) and the CERN–Columbia University–Rockefeller University collaboration at the CERN intersecting storage rings (ISR)[3] (π^0). In the region of energy and transverse momentum where the data overlap, the agreement is satisfactory. However, we would like to test the scaling behavior over as wide a range as possible of c.m. angles and energies, beyond the region covered by our experiment. For this, we consider recent data from the CERN–Università di Bologna group[4] at the ISR at smaller c.m. angles (5–20°) and higher energies, and lower-energy data from Allaby *et al.*[5] ($P_{\rm inc}$= 24 GeV/c) and Akerlof *et al.*[6] ($P_{\rm inc}$= 12.4 GeV/c). To minimize diffractive or leading-particle effects, we restrict ourselves to negative produced particles. Figure 1 shows the x_R dependence, for fixed $p_\perp$, of invariant cross sections

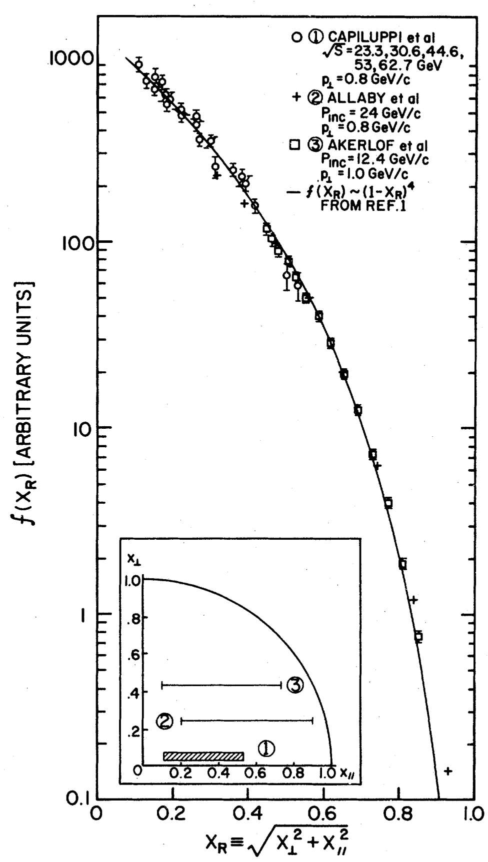

FIG. 1. Invariant cross sections for $p+p\to\pi^-+$ anything plotted versus x_R for fixed $p_\perp$. The solid curve represents the x_R dependence found in our experiment. Arbitrary normalization shifts have been made to illustrate the agreement in the x_R dependence. The inset shows the values of $x_\perp \equiv p_\perp/p_{\max}{}^*$ and $x_\parallel \equiv p_\parallel{}^*/p_{\max}{}^*$ covered by the three sets of data points.

for $p+p\to\pi^-+$anything. The open circles represent the ISR data at $p_\perp$= 0.8 GeV/c and show a dependence on x_R in good agreement with that found in our π^0 results, shown as the solid curve (arbitrarily normalized to pass through the points at small x_R). This agreement is rather remarkable considering the different kinematic regions covered by the two sets of data. We have extended the comparison to include the lower-energy data from Allaby *et al.* at $p_\perp$= 0.8 GeV/c and from Akerlof *et al.* at $p_\perp$= 1.0 GeV/c, which cover a wide range of c.m. angles and extend rather close to the kinematic boundary x_R= 1 (see inset in Fig. 1). The results of Akerlof *et al.*, when scaled from $p_\perp$= 1.0 GeV/c to $p_\perp$= 0.8 GeV/c by using Eq. (4), agree very well in absolute normalization with the data of Allaby *et al.* at $p_\perp$= 0.8 GeV/c. However, both must be scaled down by about 40% to line up with the ISR points in the region of x_R where there is overlap (this could be due to a slight deviation from scaling over the enormous energy spread involved, or to experimental normalization differences). When this is done, as shown in Fig. 1, it is apparent that the x_R dependence is very similar for all four sets of data. It therefore appears that a single universal scaling expression is consistent with inclusive pion data from pp collisions, independent of c.m. angle from 5 to 110° and for $25 \lesssim s \lesssim 4000$, if the data are expressed in terms of the variable x_R.

Thus encouraged, we are led to ask whether a similar universality exists for K^- and $\bar{p}$ data. The Chicago-Princeton[2] results taken at NAL near 90° in the c.m. system can be compared with measurements taken at very-small c.m. angles by the CERN-Bologna group.[4] Since in the NAL data both $p_\perp$ and x_R are varying, we assume that the intrinsic $p_\perp$ dependences $g(p_\perp)$ for π^-, K^-, and $\bar{p}$ are the same for large enough $p_\perp$. We then divide out the $p_\perp$ dependence in the NAL data by comparing particle ratios K^-/π^- and $\bar{p}/\pi^-$ between the two experiments. This comparison is shown in Fig. 2. The upper set of points in Fig. 2(a) shows the K^-/π^- ratios measured by the Chicago-Princeton experiment for $p_\perp > 3$ GeV/c. The ratio K^-/π^- can be seen to fall with increasing x_R. This means, given the assumption of similar forms of $g(p_\perp)$ for K^- and π^-, that the K^- radial scaling function has a more rapid x_R dependence than that for the π^- mesons. The CERN-Bologna data were taken at a fixed $p_\perp$ of 0.8 GeV/c and are shown as the lower set of points in Fig. 2(a). Although the errors on the data are large it appears that the ratio K^-/π^- again decreases with

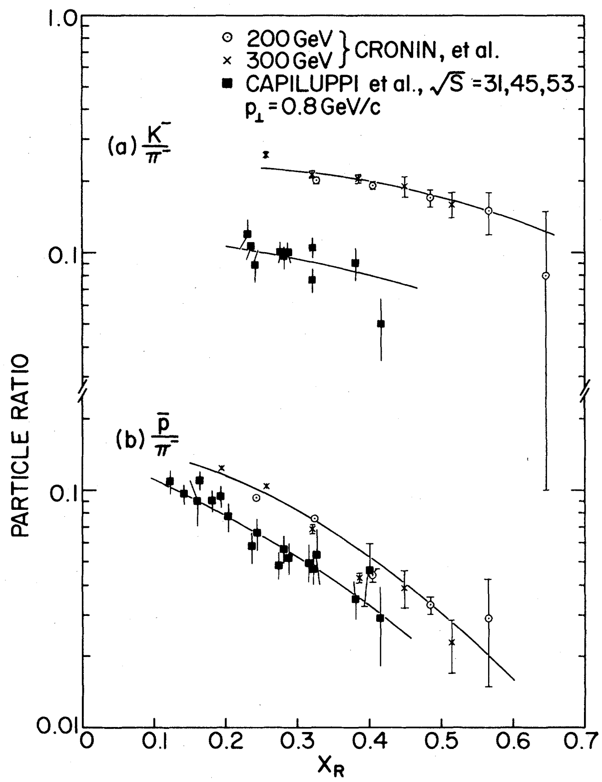

FIG. 2. Particle ratios K^-/π^- and $\bar{p}/\pi^-$ plotted versus x_R for data from Refs. 2 and 4. The curves were generated by Eqs. (5)–(7).

increasing x_R. In fact a comparison of the two sets of data, one taken at 90° and the other at very small angles in the c.m. system, shows that they are consistent with being parallel and therefore with having the same K^- radial scaling function $f_{K^-}(x_R)$. The magnitudes of the ratios K^-/π^- from the two experiments are different, but this difference may arise from the low $p_\perp$ value of the ISR data. A simple mechanism for describing this difference is to assume the form of Eq. (4) with a species-dependent mass term. To illustrate this, curves are shown in Fig. 2(a) which are good fits to the observed K^-/π^- ratios. The functions used to generate the curves are

$$E\frac{d^3\sigma}{dp^3}(\pi^-)=N(p_\perp^2+0.86)^{-4.5}(1-x_R)^4 \tag{5}$$

and

$$E\frac{d^3\sigma}{dp^3}(K^-)=0.36N(p_\perp^2+1.22)^{-4.5}(1-x_R)^5, \tag{6}$$

where N is an overall normalization constant. Thus a species dependence in the mass term in $g(p_\perp)$ can significantly affect the magnitude of the particle ratio at small transverse momentum.

Figure 2(b) shows the same comparison for the $\bar{p}/\pi^-$ ratios measured at 90° by the Chicago-Princeton group at NAL and at small angles by the CERN-Bologna group at the ISR. In this case the data from the two experiments allow a more detailed comparison. The ratio $\bar{p}/\pi^-$ at 90° can be seen to fall rapidly with increasing x_R. This means, given the assumption of similar forms of $g(p_\perp)$ for $\bar{p}$ and π^-, that the $\bar{p}$ radial scaling function has a much more rapid x_R dependence than that for π^- mesons. The small-angle ISR $\bar{p}/\pi^-$ data are strikingly parallel to the large-angle NAL data. Thus both sets of data are consistent with having the same $\bar{p}$ radial scaling function $f_{\bar{p}}(x_R)$.

Once again the magnitudes of the ratios $\bar{p}/\pi^-$ from the two experiments are different. As before we ascribe this to a species dependence in the mass term of $g(p_\perp)$ in Eq. (4). The curves shown in Fig. 2(b) were generated from Eq. (5) for the π^- cross section and the following expression for $\bar{p}$ production:

$$E\frac{d^3\sigma}{dp^3}(\bar{p})=0.26N(p_\perp^2+1.04)^{-4.5}(1-x_R)^7. \tag{7}$$

In conclusion, we have shown that when the invariant cross sections are expressed in terms of a new scaling variable x_R, available data from the Argonne National Laboratory zero-gradient synchrotron, the CERN proton synchrotron, NAL, and the ISR on inclusive π^0, π^-, K^-, and $\bar{p}$ production are consistent with (a) a factorized form:

$$E\,d^3\sigma(\pi)/dp^3=f_\pi(x_R)g_\pi(p_\perp),$$
$$E\,d^3\sigma(K^-)/dp^3=f_{K^-}(x_R)g_{K^-}(p_\perp),$$
$$E\,d^3\sigma(\bar{p})/dp^3=f_{\bar{p}}(x_R)g_{\bar{p}}(p_\perp),$$

where (b) the intrinsic $p_\perp$ dependences $g(p_\perp)$ for $\pi^{0,-}$, K^-, and $\bar{p}$ are the same for large enough $p_\perp$ i.e.

$$g_\pi(p_\perp)\propto g_{K^-}(p_\perp)\propto g_{\bar{p}}(p_\perp), \quad p_\perp\gg m_i,$$

and (c) the radial scaling functions are given approximately by

$$f_\pi(x_R)=(1-x_R)^4,$$
$$f_{K^-}(x_R)=(1-x_R)^5,$$
$$f_{\bar{p}}(x_R)=(1-x_R)^7.$$

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Minimal Three-Body Scattering Theory

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We show that implementation via unitarity and analyticity of quantum mechanical constraints due to pair final-state interactions in three-body states leads to the simplest form of linear scattering integral equations usually used for the three-body problem. This form of the separable interaction equation is therefore the minimal equation compatible with these general constraints of quantum mechanics.

The general problem of final-state interactions in three-body final states is very poorly understood in spite of a wide variety of remarkably successful three-body calculations in particular systems. In this note we show that the dynamical scheme used in most of these calculations is essentially the minimal embodiment of the general constraints of quantum mechanics required by these final-state interactions. This provides a bridge between the usual formalism, based on linear off-shell scattering integral equations with separable interactions, and the general constraints of quantum mechanics expressed through on-shell unitarity and analyticity. Apparently much of the structure in three-body scattering successfully accounted for in these calculations is little more than the manifestation of the nearly kinematic constraints due to two-body final-state interactions. This may account for the great success of the simple dynamical calculations, as well as stressing again that they give little detailed dynamical insight. The technical difficulty of doing this calculation also shows that implementing final-state unitarity and analyticity in the three-body problem is complicated.

In a recent Letter,[1] referred to hereafter as A-A, we showed that unitarity applied to three-body final states implies singularities in the quasi-two-body amplitudes usually assumed to be slowly varying in the analysis of three-body final states. The strength of these singular parts is linearly related back to these same amplitudes —a manifestation of important coherence in the final state. In this Letter we exploit the fact that unitarity provides the discontinuity across the pair subenergy cut to write a dispersion relation for the amplitude. Because of the linear relation of the discontinuity to the amplitude itself, this dispersion relation yields a linear scattering integral equation which turns out to be the simplest version of the separable interaction three-body equation.

Consider an amplitude $T_{2,3}$ for going from a two-body state to a three-body state. Make the standard isobar, Faddeev, sequential-decay, or multiple-scattering decomposition of $T_{2,3}$ into a sum of terms depending on which pair interacts last:

$$T_{2,3}=\sum_i f_i \tau_i \tag{1}$$

where τ_i is the two-body scattering amplitude of the j-k pair ($i \neq j \neq k$) and f_i is its coefficient in $T_{2,3}$. Usually τ_i is taken to be dominated by a particular partial wave l_i, and then a factor of q^{-l_i} (where q_i is the j-k relative momentum) is explicitly removed from f_i. In A-A we showed