High-Density Phase Transitions in Gauge Theories

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By the introduction of a chemical potential for an absolutely conserved fermion quantum number, we determine the onset of a phase transition in a general renormalizable, finite-temperature field theory with spontaneously broken symmetries. We also briefly investigate the possible physical existence of such a phase transition in black holes and in the early universe.

Recent investigations[1,2] on phase-transition phenomena in a renormalizable quantum field theory indicate that finite-temperature effects can restore a symmetry which is broken at zero temperature. This occurs through the generation of a temperature-dependent scalar boson mass (or inverse correlation length) which, for sufficiently high temperatures, may vanish. The question we address is whether the same mechanism is operative for a large chemical potential due to an absolutely conserved fermion quantum number. Alternatively expressed, can a large fermion number density effect a phase transition?

In the framework of a weakly interacting, renormalizable field theory characterized by a coupling constant $e \ll 1$, the effective boson mass can only vanish if powers of the temperature θ or chemical potential μ can compensate for powers of the coupling e. Weinberg has shown[2] that when $\overline{\mu} = \mu/\theta \ll 1$, the leading terms in any graph for e small and θ large are those in which all loops beyond the lowest order are quadratically divergent. The finite part of such a loop contributes a factor proportional to $e^2\theta^2$, so that the critical temperature is attained when $e^2\theta^2$ is of the order of the square of the renormalized mass at zero temperature, $\mathfrak{M}_R^2$, i.e., $\theta_{crit} \approx \mathfrak{M}_R/e$. In typical gauge theories of the weak and electromagnetic interactions,[3] $\mathfrak{M}_R/e$ is of the order of $G_F^{-1/2}$ so that $\theta_{crit} \approx 300$ GeV $\approx 10^{15}$ °K.

We are particularly interested in isolating the leading terms to any graph for e small and μ large, and $\overline{\mu} \gg 1$. Considering a loop expansion[2,4] we have a factor of e^{2L} in a graph with L loops. To determine the powers of μ contributed by each loop, we concentrate on a single loop with superficial degree of divergence D. We rescale by θ all dimensional factors in the single-loop integration so that the whole loop takes the form

$$\theta^D I(p_{ext}/\theta, \omega_{ext}/\theta, m_{int}/\theta, \overline{\mu}), \tag{1}$$

where p_{ext} and ω_{ext} represent the set of external momenta and energies, and m_{int} represents the internal masses. Now the single loop integration will contain[5] a Fermi distribution function in the integrand of the form

$$F\left(\frac{\mathcal{E}}{\theta}, \overline{\mu}\right) = \frac{1}{e^{(\mathcal{E}-\mu)/\theta}+1} + \frac{1}{e^{(\mathcal{E}+\mu)/\theta}+1}, \tag{2}$$

where $\mathcal{E} = (p^2+m^2)^{1/2}$ and m is an eigenvalue of the the fermion mass matrix. Since $\overline{\mu} \gg 1$, the second term in (2) will act as an exponentially damping factor in the loop momentum p, while the first term will begin providing an exponential falloff in p when $\mathcal{E} \gtrsim \mu$, i.e., above the Fermi energy. We are primarily interested in μ large (≈ 300 GeV) and thus much greater than m, so that the integral in (1) is effectively cut off when $p \sim \mu$. Thus

$$\theta^D I(p_{ext}/\theta, \omega_{ext}/\theta, m_{int}/\theta, \overline{\mu}) \sim \theta^D \overline{\mu}^D = \mu^D. \tag{3}$$

Since we are considering renormalizable field theories, the largest value of D that concerns us is $D=2$. In addition, all quadratic divergences can be eliminated by a renormalization of the boson self-mass (with the assumption that there are no gauge-invariant scalar fields). Thus, the effective boson self-mass will contain a term of the form $e^2\mu^2$, and we anticipate a phase transition when $\mu_{crit} \approx \mathfrak{M}_R/e$. Again, in typical gauge theories of the weak and electromagnetic interactions[3] this transition would occur for $\mu_{crit} \approx 300$ GeV. Dimensionally, for $\overline{\mu} \gg 1$, we expect $\mu \sim n^{1/3}$, where n is the fermion number density, so that the critical fermion number density would be of the order $n_{crit} \sim 10^{48}/\text{cm}^3$.

These anticipated results can be explicitly verified by calculating the leading (in μ) one-loop contributions to the boson self-mass. We consider a general renormalizable Lagrangian[6] which possesses a gauge invariance with respect to some compact semisimple Lie group G:

$$\mathcal{L} = -\tfrac{1}{4}F_{\alpha\mu\nu}F_\alpha{}^{\mu\nu} - \tfrac{1}{2}(D_\mu\varphi)_i(D^\mu\varphi)_i - \overline{\psi}\gamma^\mu D_\mu\psi - \overline{\psi}m_0\psi - P(\varphi) - \overline{\psi}\Gamma_i\psi\varphi, \tag{4}$$

where $F_{\alpha\mu\nu}$ is the gauge-covariant curl of a set of Hermitian gauge fields, $(D_\mu\varphi)_i$ is the gauge-covariant derivative of a multiplet of real, spin-0 fields, $D_\mu\psi$ is the gauge-covariant derivative of a multiplet of spin-$\frac{1}{2}$ fields, m_0 is the Hermitian and gauge-preserving bare fermion mass matrix, $P(\varphi)$ is a real fourth-order gauge-invariant polynomial in φ, and Γ_i is the set of Hermitian and gauge-covariant Yukawa coupling matrices. The gauge invariance of the Lagrangian is broken by allowing the scalar fields φ_i to develop spontaneously a nonvanishing vacuum expectation value.

We consider this Lagrangian at finite temperature and incorporate a fermion chemical potential as a Lagrange multiplier of the number density η given by

$$\eta = \psi^\dagger\psi = \overline{\psi}\gamma_4\psi. \tag{5}$$

Thus, the chemical potential μ will alter the fourth component of the fermion propagator, which, at finite temperature, implies the replacement of the discrete energy $\mathcal{E}_n = (2n+1)\pi\theta$, n an integer, by $\mathcal{E}_n - i\mu$.

As previously mentioned, the leading chemical-potential effects, as well as all quadratic divergences, can be absorbed into a redefinition of the quadratic terms in the potential $P(\varphi)$ by defining an effective potential[2]

$$P_{\rm eff}(\varphi) = P(\varphi) + \tfrac{1}{2}Q_{ij}(\mu)\varphi_i\varphi_j \tag{6}$$

and by adding a compensating counterterm to the Lagrangian,

$$\delta\mathcal{L}' = \tfrac{1}{2}Q_{ij}(\mu)\varphi_i\varphi_j. \tag{7}$$

Here, $Q_{ij}(\mu)$ is some chemical-potential-dependent, quadratically divergent matrix which contributes to the formation of an effective scalar boson mass. After shifting the scalar fields by their zeroth-order vacuum expectation value λ, we note that the contributions to the quadratic divergences in any renormalizable theory come from tadpoles T_i and the boson self-energy Π_{ij}. Furthermore, $Q_{ij}(\mu)$ is determined by requiring that the counterterms provided by (7) cancel these quadratic divergences as well as those terms proportional to $e^2\mu^2$. It is the latter terms that we wish to isolate, since the former terms can be absorbed into a temperature- and chemical-potential-independent renormalized scalar mass $\mathfrak{M}_R$. But the chemical potential enters the calculation only through the fermion propagators. Thus, the graphs of interest are the fermion-loop tadpole (Fig. 1) and the fermion-loop contributions to the boson self-energy (Fig. 2).

The fermion-loop tadpole at finite temperature and chemical potential is given by

$$T_i{}^\psi = i(2\pi)^4\mathrm{Tr}\{\Gamma_i\gamma_4\Gamma_j\gamma_4\}\lambda_j I_F(\theta,\mu), \tag{8}$$

where

$$I_F(\theta,\mu) = \theta\sum_{n=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\mathcal{E}^2 + (\mathcal{E}_n - i\mu)^2} \tag{9}$$

and $\mathcal{E}^2 = p^2 + m^2$. Performing the summation over n and renormalizing at $\theta = \mu = 0$, we find

$$I_F{}^R(\theta,\mu) = -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\mathcal{E}}\,F\left(\frac{\mathcal{E}}{\theta},\overline{\mu}\right), \tag{10}$$

where the Fermi function F is defined in (2). In the limiting case of n large and $\overline{\mu} \gg 1$, we have

$$I_F{}^R(\theta,\mu) \approx -\frac{\theta^2}{4\pi^2}\int_0^\infty dx\,x\left(\frac{1}{e^{x+\overline{\mu}}+1} + \frac{1}{e^{x-\overline{\mu}}+1}\right). \tag{11}$$

This integral can be done exactly to yield

$$I_F{}^R(\theta,\mu) \simeq -\tfrac{1}{24}\theta^2[1 + (3/\pi^2)\overline{\mu}^2]. \tag{12}$$

But the counterterm (7) contributes a term

$$\delta T_i{}^\psi = i(2\pi)^4 Q_{ij}(\mu)\lambda_j, \tag{13}$$

so that we have for the finite part of $Q_{ij}(\mu)$ in the limit $\overline{\mu} \gg 1$

$$Q_{ij}{}^{\rm fin}(\mu) = (\mu^2/8\pi^2)\mathrm{Tr}\{\Gamma_i\gamma_4\Gamma_j\gamma_4\}. \tag{14}$$

Thus, the effective scalar boson mass matrix is

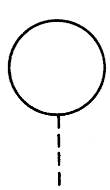

FIG. 1. Fermion-loop tadpole. Dashed line, scalar; solid line, spinor.

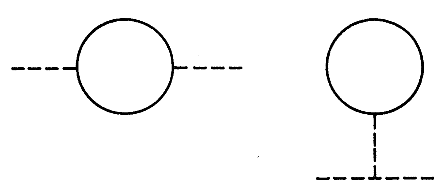

FIG. 2. Fermion-loop contributions to the boson self-energy.

given by

$$[\mathfrak{M}^2(\mu)]_{ij} = [\mathfrak{M}_R{}^2]_{ij} + (\mu^2/8\pi^2)\mathrm{Tr}\{\Gamma_i\gamma_4\Gamma_j\gamma_4\}, \qquad (15)$$

where the chemical potential provides a positive-definite contribution. The vanishing of $\mathfrak{M}^2(\mu)$ determines $\mu_{\rm crit}$ in the one-loop approximation. In the case of a spontaneously broken symmetry where $\mathfrak{M}_R{}^2 < 0$, we have $\mu_{\rm crit} \sim \mathfrak{M}_R/e$ since the Yukawa coupling matrices are each proportional to e.

This result can be checked by noting that the counterterms provided by (7) also cancel the leading fermion-loop contributions to the scalar boson self-energy (see Fig. 2) given by

$$\Pi_{ij}{}^{\psi} = i\theta \sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr}\{\Gamma_i(i\gamma_\alpha p^\alpha + m)^{-1}\Gamma_j(i\gamma_\alpha p^\alpha + m)^{-1}\} + i f_{ijk} M_{kl}{}^{-2} \sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr}\{\Gamma_l(i\gamma_\alpha p^\alpha + m)^{-1}\}, \qquad (16)$$

where

$$p^0 = \mathcal{E}_n - i\mu, \quad M_{ij}{}^2 = \left.\frac{\partial^2 P(\varphi)}{\partial\varphi_i\,\partial\varphi_j}\right|_{\varphi=\lambda}, \text{ and } f_{ijk} = \left.\frac{\partial^3 P(\varphi)}{\partial\varphi_i\,\partial\varphi_j\,\partial\varphi_k}\right|_{\varphi=\lambda}.$$

Isolating the quadratically divergent part of (16) and the leading terms in μ, we find

$$\Pi_{ij}{}^{\psi} \simeq -Q_{ij}(\mu) + f_{ijk} M_{kl}{}^{-2} Q_{lm}(\mu)\lambda_m. \qquad (17)$$

The first term is canceled directly by the counterterm in (7), while the second term is canceled by a tadpole produced by the counterterm.

We can equivalently express our result in terms of a critical number density $n_{\rm crit}$, where n is the number of fermions minus the number of antifermions in a unit volume V:

$$n = (1/V)\int d^3x\,\eta(\vec{x}). \qquad (18)$$

Using (5) and the μ-dependent free-fermion propagator at finite temperature, we have for the renormalized (at $\theta = \mu = 0$) number density

$$n_R(\theta,\mu) \simeq \frac{\theta^3}{\pi^2}\int_0^\infty dx\, x^2\left(\frac{1}{e^{x-\bar\mu}+1} - \frac{1}{e^{x+\bar\mu}+1}\right), \qquad (19)$$

where the fermion mass is again negligible compared to μ. This integral can be done exactly and we find

$$n_R(\theta,\mu) \simeq (\theta^3/3\pi^2)(\bar\mu^3 + \pi^2\bar\mu). \qquad (20)$$

Inverting (20) yields approximately

$$\mu \simeq (3\pi^2 n_R)^{1/3}[1 + O(1/\bar\mu)], \qquad (21)$$

thus confirming our earlier dimensional argument leading to the estimate of the critical fermion number density of $10^{48}/\mathrm{cm}^3$. Densities of this magnitude or greater are considered in the evolution of black holes[7] and of the universe.[8]

A gravitationally collapsing star with mass M and original radius R_0 will theoretically reach a point of infinite density in a finite proper time given by[7]

$$\tau = \pi(R_0{}^3/8M)^{1/2}, \qquad (22)$$

i.e., the time for total collapse as measured by a clock on the surface of the collapsing star depends only on the initial density. Since a typical stellar mass is $\approx 10^{33}$ g (which is of the order of the Chandrasekhar and Oppenheimer-Volkoff limits),[9] there may be up to 10^{57} baryons. Then, the critical density would occur when the radius was $\approx 10^3$ cm and when the proper time was given essentially by (22).

It has been noted[1] that a phase transition which produces massless vector particles may lead to anomalously great repulsive forces between the particles of the system. Whether this force is sufficient to reverse the collapse of the star has yet to be determined.

Similar considerations can be applied to the universe as a whole, extrapolating backwards in time and assuming the standard big-bang model with elementary particles.[10] If we take the present radius of the universe as 10^{10} light years and the density as 10^{-30} g/cm^3, we find a maximum number of baryons of 10^{78}. This would lead to a phase transition when the radius of the universe was approximately 10^{10} cm.

However, from the observed cosmic-radiation temperature and mass density, we know that the dimensionless chemical potential is small today, i.e., $\bar\mu \simeq 10^{-8}$–10^{-9}.[2,9] In addition, when $\bar\mu \ll 1$, $\bar\mu \propto 1/\sigma$, where σ is the dimensionless specific entropy per baryon. Now if we presume an adiabatic expansion of the universe, σ is constant, so that $\bar\mu$ would never grow large enough to effect a phase transition. The adiabatic assumption, though, leaves unexplained both the origin of the large specific entropy and the origin of the galaxies. An attractive solution[11-13] which could solve these problems simultaneously asserts that the viscous dissipation of anisotropies and possible inhomogeneities would serve to generate entropy but preserve "small" mass fluctuations of galactic size. Such a solution would necessarily imply a growth of the specific entropy in time or a corresponding increase of $\overline{\mu}$ extrapolated back in time, with $\overline{\mu} \gg 1$ allowed in principle.

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Inclusive π^0 Production in *pp* Collisions at 50–400 GeV/*c**

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We have measured the single-photon cross section in the reaction $p+p\to\gamma+$anything for incident proton momenta from 50 to 400 GeV/*c* and lab angles of 80, 100, and 120 mrad. It is shown that in the range $p_\perp=0.3$ to 4.3 GeV/*c*, the derived π^0 invariant cross section can be factorized into a product of two functions, one in $p_\perp$ and the other in a new scaling variable $x_R=p^*/p_{\max}^*$, where p^* is the total c.m. momentum of the π^0.

The study of the production of pions with large transverse momentum $p_\perp$ in proton-proton collisions is expected to give insight into the short-distance structure of the proton.[1-3] Great interest in this field has been stimulated by the experimental results obtained at the CERN intersecting storage rings[4-6] and at the National Accelerator Laboratory (NAL).[7,8] Büsser *et al.*[4] at the intersecting storage rings were the first to show that near 90° in the *pp* c.m. system, the large-$p_\perp$ data were consistent with a scaling behavior with respect to the variable $x_\perp=2p_\perp/\sqrt{s}$. On the other hand, it has been known for several years that the small-$p_\perp$ single-pion inclusive data exhibit scaling with respect to the variable $x_\parallel=2p_\parallel^*/\sqrt{s}$ at all c.m. angles. In this experiment, data have been obtained on single-π^0 inclusive spectra from 40 to 110° in the *pp* c.m. system and for $0.3 \leq p_\perp \leq 4.3$ GeV/*c*. It has been found that the cross sections scale when the "radial" variable $x_R=p^*/$