tions.

In summary, we have produced direct visual evidence that individual, discrete vortex lines exist as predicted by Feynman and Onsager. It appears that under our experimental conditions no stable vortex array is observed; rather the rotating He II exists in a dynamic state involving complex motion of vortex lines.

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Mass-Fluctuation Waves in Solid ^{3}He-^{4}He Mixtures*

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We give an explanation of the data of Richards, Pope, and Widom in terms of the incoherent tunneling of strongly interacting particles.

The concept of "impuritons" or "mass-fluctuation waves" was invented independently by Andreev and Lifshitz[1] and Guyer and Zane.[2] The early survey experiments by Myoshi, Cotts, Greenberg, and Richardson[3] suggested that dilute solid mixtures of ^{3}He in ^{4}He are a suitable system in which to seek evidence for these excitations.[4] Subsequently Greenberg, Thomlinson, and Richardson[5,6] reported T_1 and T_2 measurements that showed compelling evidence for ^{3}He tunneling in ^{4}He. Richards, Pope, and Widom (RPW)[7,8] have undertaken an extensive set of measurements on dilute ^{3}He in ^{4}He mixtures designed to find evidence for the coherent motion of the impurity, i.e., for the existence of ^{3}He "impuritons." These measurements include diffusion data in addition to T_1 and T_2 data. RPW have argued that a diffusion measurement gives a clear signature for coherent motion. Recently Grigor'ev and co-workers[9] have verified the diffusion measurements of RPW.

Many aspects of the data of RPW are consistent with the qualitative features expected using the "impuriton" description of the ^{3}He motion.[7,8] But there are two serious difficulties in the quantification of such a description: (1) The rate of ^{3}He-^{4}He tunneling must be taken to be about 2 orders of magnitude below what is regarded as a plausible theoretical value,[10] and (2) the interaction between ^{3}He impurities must be taken to be about 3 orders of magnitude below what is regarded as a plausible theoretical value.[4] This paper provides an explanation of the measurements of RPW and of Grigor'ev and co-workers in terms of ^{3}He tunneling in the presence of a strong ^{3}He-^{3}He interaction.

The theory of ^{3}He motion through solid ^{4}He that we discuss depends in a crucial way on the ^{3}He-^{3}He interaction. To learn about this interaction we reformulate the mixture problem for general concentration in order to derive the ^{3}He-^{3}He interaction in a framework that permits us to determine its magnitude from first principles and to determine its relationship to the phase-separation problem. We take a mixture of ^{3}He in ^{4}He at concentration $x_3 = N_3/(N_3+N_4)$ to be described by $\mathcal{H} = \overline{\mathcal{H}} + \mathcal{H}'$, where

$$\overline{\mathcal{H}} = \sum_R \tfrac{1}{2} p_R^2 \langle m^{-1}\rangle + V, \tag{1}$$

$$\mathcal{H}' = \sum_R \tfrac{1}{2} p_R^2 (m_R^{-1} - \langle m^{-1}\rangle) = \sum_R \Delta \overline{K}(R), \tag{2}$$

with $\langle m^{-1}\rangle = x_3 m_3^{-1} + (1-x_3) m_4^{-1}$ and V the pairwise atom-atom interaction between atoms localized on lattice sites R. We solve the ground-state problem for the average crystal described by $\overline{\mathcal{H}}$ at various concentrations, employing a typical quantum-crystal procedure,[10] and find a set of low-lying single-particle states and the corresponding t-matrix elements required to second quantize $\mathcal{H}$. Using the states and matrix elements appropriate to concentrations x_3, we obtain the Hamiltonian in the form

$$\mathcal{H}(x_3) = E_0(x_3) + \mathcal{H}_p + \mathcal{H}_I, \tag{3}$$

where $E_0(x_3)$ is the ground-state energy of the average crystal, $\mathcal{H}_p$ is the phonon Hamiltonian, and

$$\mathcal{H}_I = -\sum_{RR'} \Delta \overline{K}(R)_{02} \mathfrak{D}(RR') \Delta \overline{K}(R')_{02}. \tag{4}$$

Here

$$\Delta \overline{K}(R)_{02} = -\tfrac{1}{2}\hbar^2 (\nabla^2)_{02} (m_R^{-1} - \langle m^{-1}\rangle)$$

and

$$\mathfrak{D}(RR') = \sum_{SS'} t(R,S)_{02,01} (\Delta\epsilon_{20})^{-1} D(SS') (\Delta\epsilon_{20})^{-1} t(S'R')_{01,02}. \tag{5}$$

We have used three low-lying single-particle states: the ground state 0, the displacement state 1, and the width-fluctuation state 2.[11] The t-matrix element $t(R,S)_{02,01}$ is the coupling of a width fluctuation at R [due to the deviation of $m^{-1}(R)$ from $\langle m^{-1}\rangle$] to a displacement fluctuation at S; it is proportional to the cubic anharmonicity. $\Delta\epsilon_{20} = \epsilon_2 - \epsilon_0$; $D(SS')$ is the inverse of the dynamical matrix. We evaluate $D(SS')$ using elastic continuum theory[12] and determine $\Delta\overline{K}(R)_{02}(\Delta\epsilon_{20})^{-1} t(RR')_{02,01}$ by comparison of the theoretical near-neighbor displacement $u_{R'} = x_{01}(\Delta\epsilon_{10})^{-1} t(R'R)(\Delta\epsilon_{20})^{-1}\Delta K(R)_{02}$ with the numerical results of Mullin[13] and Glyde.[14] Altogether we obtain

$$\mathcal{H}_I = -V_0 \sum_{RR'} (\Delta/|\vec{R}-\vec{R}'|)^3 [n_3(R) - x_3][n_3(R') - x_3], \tag{6}$$

with $V_0/k_B \approx 10^{-2}$ K at 21.0 cm^3/mole and Δ the interparticle spacing. Phase separation is determined entirely by $E_0(x_3)$ and involves basic energies quite different in source and magnitude from the energies associated with $\mathcal{H}_I$.

Our primary concern is to determine the effect of $\mathcal{H}_I$ on the propagation of a ^{3}He impurity through ^{4}He. It is important to note that the rate of ^{3}He-^{4}He tunneling, $J_{34}/\hbar$, is expected to be at most comparable with[4] $|J_{33}|/\hbar \approx 5\times 10^{-5}$ K so that we have $|J_{34}|/V_0 < 10^{-2}$. We take $x_3 < 10^{-2}$ as defining the low-concentration region which we further divide into a *dilute region,* $\overline{x} < x_3 < 10^{-2}$, and an *"impuriton" region,* $0 < x_3 < \overline{x}$.

The primary effect of the interaction in $\mathcal{H}_I$ is to hinder the tunneling of a ^{3}He-^{4}He pair. In the initial configuration, ^{3}He at R and ^{4}He at R', the potential energy of the pair is

$$V(RR') = -V_0 \sum_{S\neq(RR')} \left[\left(\frac{\Delta}{|\vec{R}-\vec{S}|}\right)^3 (1-x_3) - \left(\frac{\Delta}{|\vec{R}'-\vec{S}|}\right)^3 x_3\right][n_3(S) - x_3], \tag{7}$$

whereas after the tunneling process we have

$$V(R'R) = +V_0 \sum_{S\neq(RR')} \left[\left(\frac{\Delta}{|\vec{R}-\vec{S}|} \right)^3 x_3 - \left(\frac{\Delta}{|\vec{R}'-\vec{S}|} \right)^3 (1-x_3) \right] [n_3(S) - x_3], \tag{8}$$

or a net change in potential energy of

$$\Delta V(RR') = -V_0 \sum_{S\neq(RR')} \left[\left(\frac{\Delta}{|\vec{R}'-\vec{S}|} \right)^3 - \left(\frac{\Delta}{|\vec{R}-\vec{S}|} \right)^3 \right] [n_3(S) - x_3)]. \tag{9}$$

If $\Delta V(RR')$, proportional to the gradient of $\mathcal{H}_I$, is small compared to J_{34}, the tunneling process occurs easily; if $\Delta V(RR')$ is larger than J_{34}, the tunneling process is hindered. In this latter circumstance energy can be conserved in the transition only if the tunneling is accompanied by the absorption or emission of a phonon. But a sequence of phonon-assisted steps is incoherent. Thus we expect that a ^{3}He impurity will propagate as an impuriton in regions of space where the gradient of $\mathcal{H}_I$ is small. At the very lowest concentrations a ^{3}He impurity propagates coherently as an impuriton until it approaches within $\bar{r}$ of a second impurity, where $\bar{r}$ is defined by $J_{34} = \Delta |\nabla \mathcal{H}_I(\bar{r})|$. Thus $\pi\bar{r}^2 \approx \pi\Delta^2(3V_0/J_{34})^{1/2}$ is the cross section for impuriton-impuriton scattering and the corresponding mean free path is

$$\lambda = (\Delta/\pi x_3)(J_{34}/3V_0)^{1/2}. \tag{10}$$

So in the "impuriton" region the diffusion constant is

$$D_I \approx \pi^{-1}\Delta^2 J_{34}(J_{34}/3V_0)^{1/2}(1/x_3). \tag{11}$$

As the concentration is increased through the value $\bar{x}$ to concentrations at which a typical ^{3}He atom is continually in interaction with its neighbors, the calculation of D proceeds differently than above. At $x_3 > \bar{x}$ the Kubo formula[15] for D can be manipulated to yield

$$D_D = z(\pi/6)\Delta^2 J_{34} P(0), \tag{12}$$

where $P(0)$ is the probability that $\Delta V(RR')$ is of order $2|J_{34}|$. This equation describes diffusion over steps of length Δ at the rate $J_{34}P(0)$. This rate involves two factors: J_{34}, an attempt frequency (for the transition $RR' \rightarrow R'R$), and $P(0)$, the probability that the transition can go (that the energy difference between the two arrangements if of order J_{34} and the transition $RR' \rightarrow R'R$ is energy conserving). We have made a Monte Carlo study of the spectrum of $\Delta V(RR')$, Eq. (9), and of $P(0)$ as a function of concentration for various values of the ratio $|J_{34}|/V_0$. For the choice $J_{34} = 0.5|J_{33}| = 2.5\times10^{-5}$ K and $V_0 = 10^{-2}$ K we find the results for D shown in Fig. 1. The spectrum of $\Delta V(RR')$ is strongly peaked near zero at low concentrations, $P(0)$ approaches 1 as x_3 gets small, and $P(0)$ is greater than $\frac{1}{2}$ at $x_3 = 10^{-3}$ (see Fig. 2). The behavior we find for D_D is similar to that described by Landesman and Winter.[16] At $x_3 < 10^{-3} \equiv \bar{x}$ a ^{3}He has a better than 50% probability of being in a low-gradient region. Thus we expect impuriton behavior at $x_3 \ll \bar{x}$. The results of a generalization of this procedure to include multistep processes are also shown in Figs. 1 and 2.

Can this discussion of the D data also lead to a satisfactory explanation of the T_1 and T_2 data? We begin by exhibiting an expression for T_2 as $\omega_0 \rightarrow 0$:

$$\frac{1}{T_2} = M_2(1)x_3 \left[\sum_{R'\neq(R)} \left(\frac{\Delta}{|\vec{R}-\vec{R}'|} \right)^6 \int_0^{+\infty} dt \langle\langle n_3(R,0)n_3(R',0)n_3(R,t)n_3(R',t)\rangle\rangle \Big/ \sum_{R'\neq(R)} \left(\frac{\Delta}{|\vec{R}-\vec{R}'|} \right)^6 \right]. \tag{13}$$

This formula exhibits the most important feature of T_1 and T_2 data. Because the dipolar field falls off so rapidly, r^{-6}, the pairs of particles that make important contributions to T_1 and T_2 are always well within the interaction range $\bar{r}$ where the particle motions are incoherent. Thus it is *impossible* to see impuriton behavior in T_1 and T_2 data. To carry out a calculation of T_2 we approximate the correlation function in Eq. (13) by assuming that the particles at R and R' move separately; i.e.,

$$\langle\langle n(r,0)n(R',0)n(R,t)n(R',t)\rangle\rangle \approx \langle\langle (R,0)n(R,t)\rangle\rangle_R \langle\langle n(R',0)n(R',t)\rangle\rangle_R;$$

here the subscript on $\langle\langle n(R,0)n(R,t)\rangle\rangle_{R'}$ means that we calculate the correlation function for a particle at R given that there is a ^{3}He spectator particle fixed at R'. Following an argument similar to that

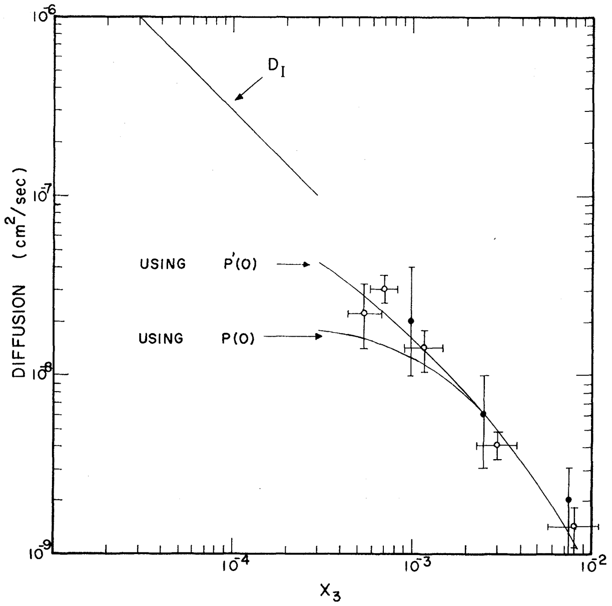

FIG. 1. Diffusion constant as a function of concentration. In the range $5\times10^{-4}<x_3<2\times10^{-2}$ the value of D given by Eq. (12) with $V_0=10^{-2}$ and $J_{34}=2.5\times10^{-5}$ is shown as the lower solid line. The data of Richards, Pope, and Widom and Grigor'ev and co-workers are indicated as open and closed circles, respectively. At low concentration we show the value of D in the "impuriton" region calculated from Eq. (11) using V_0 and J_{34} as above. We also show the results of an improvement in the theory which includes the possibility of particle motion over more than one step, the curve labeled $P'(0)$. These multistep processes are important at lower concentrations, $x_3\lesssim10^{-3}$.

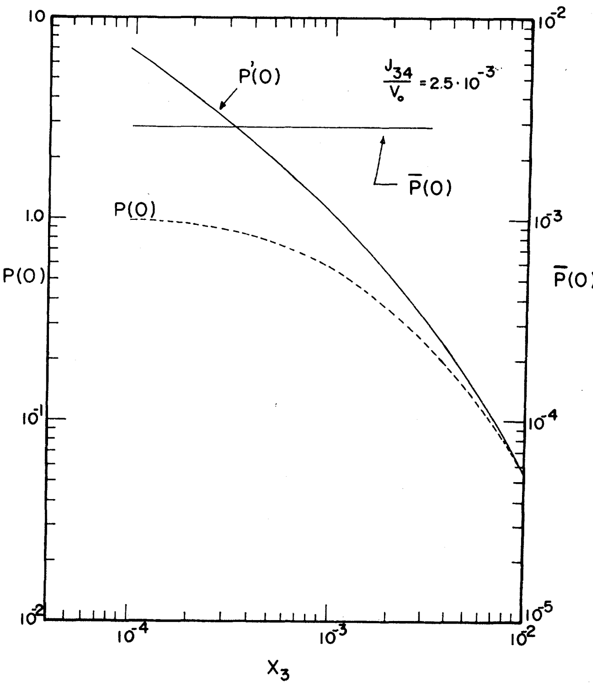

FIG. 2. $P(0)$ and $\bar{P}(0)$ as a function of concentration. $P(0)$ is read from the left-hand scale and $\bar{P}(0)$ is read from the right-hand scale. Monte Carlo studies of Eq. (4) permit the calculation of $P(0)$ and $\bar{P}(0)$ as a function of concentration. The probability $P(0)$ approaches 1 at $x_3\ll10^{-3}$; it has value $\frac{1}{2}$ at $x_3\approx10^{-3}$. Thus we take 10^{-3} as the lower edge of the dilute region. The probability $\bar{P}(0)$ is essentially concentration independent for $x_3<10^{-2}$. We also show $P'(0)$, a generalization of $P(0)$ to processes of more than one step.

leading to Eq. (12) for the diffusion constant, we obtain

$$\frac{1}{T_2(0)}=M_2(1)x_3\frac{1}{\pi J_{34}\bar{P}(0)}$$
$$\equiv M_2(1)x_3\frac{1}{W_2(x_3)_{RR'}},\qquad(14)$$

where $\bar{P}(0)$ is the weighted average of $P_{R''}(0)$, the probability that $\Delta V(RR')$ is of order J_{34} given that there is a spectator ^{3}He at R''. The weighting factor is $|R-R''|^{-6}$. We have made Monte Carlo studies of $\bar{P}(0)$ and $W_2(x_3)_{RR'}$ for the choice $J_{34}/V_0=2.5\times10^{-3}$ for various concentrations. For $x_3<10^{-2}$ we find $W_2(x_3)_{RR'}$ to be essentially independent of x_3 and of order 4×10^4 rad/sec (see Fig. 2). Thus in both the dilute and impuriton regions we have $T_2^{-1}\propto x_3$. We state this result for T_2 in terms of the ratio of D to T_2. We find from Eqs. (12) and (14) that

$$D/T_2=M_2(1)\Delta^2x_3[2P(0)/\bar{P}(0)].\qquad(15)$$

Using the results from our Monte Carlo studies of $P(0)$ and $\bar{P}(0)$ shown in Fig. 2 $[P'(0)\approx10^{-3}x_3^{-1}$, $\bar{P}(0)\approx3\times10^{-3}]$, we find

$$D/T_2\approx0.7M_2(1)\Delta^2.\qquad(16)$$

This result is in reasonable agreement with the observations of RPW who find that this ratio (for their experiments) is the same as for gaseous ^{3}He.

We find that an interaction model, applied in the dilute-concentration region, can satisfactorily explain both the D and T_2 data of Richards, Pope, and Widom. Further this model makes use of plausible values of the parameters required to describe the system. For $V_0\approx10^{-2}$ K we require $J_{34}\approx\frac{1}{2}|J_{33}|$. We believe that a more satisfactory explanation of the behavior of ^{3}He impurities in the dilute-concentration region $\bar{x}\leq x_3\leq10^{-2}$ is given by this model than by the "impuriton" model. We set an upper limit on the concentration at which impuriton behavior occurs, $x_3 \ll \bar{x} = 10^{-3}$. The interaction model yields a simple explanation of the strong volume dependence of the basic rate observed by Greenberg, Thomlinson, and Richardson and by Grigor'ev and coworkers.

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Beam-Plasma Instability in a Nonuniform Plasma

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The behavior of the beam-plasma instability in a nonuniform plasma depends only on the dimensionless quantity $\lambda \sim k_i^{-2}\partial k/\partial x$, where k_i is the spatial growth rate in the homogeneous plasma. If $|\lambda| \geq 1$ the instability is quenched.

The beam-plasma instability for a weak beam in a uniform plasma is the result of resonant coupling between the beam and an eigenmode of the plasma. If the plasma is nonuniform, because of density or magnetic field gradients, the plasma eigenmode characteristics are spatially dependent and one may expect a detuning of the beam plasma resonance along the gradient. Indeed, this process is similar to that of parametric instabilities in nonuniform media where density gradients result in increased thresholds and reduced growths of the instabilities.[1] Gradients are inevitable in experimental arrangements, and therefore it is important to understand their consequences for instabilities. Indeed, there is some experimental evidence of localization and thresholds of the beam-plasma instability due to density gradients along the direction of beam propagation.[2,3]

The problem of quasilinear relaxation of an ultrarelativistic beam with a large velocity spread in an inhomogeneous plasma has been treated by Breizman and Ryutov[4] and the behavior of a wave packet in a cold nonuniform plasma with a cold beam by Vianna and Bers.[5] In the following we investigate the electrostatic instability of a cold electron beam propagating along a weak gradient in the plasma. The gradient can be one of density, temperature, magnetic field, etc.

The linearized equations of motion for an elec-