# Asymptotic Behavior of Non-Abelian Gauge Theories to Two-Loop Order*

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We have calculated the value of the Callan-Symanzik $\beta$ function to order $g^5$ for non-Abelian gauge theories with fermions. We discuss internal consistency of the calculation, and consider the approach to the aymptotic energy range in such theories.

There has recently been much interest in non-Abelian gauge theories of the strong interactions. Renormalization-group parameters have been calculated to lowest order,[1] and a subclass of these theories has been found to asymptotically free.[2] The value of $\beta$ in $g^5$ order is interesting for study of the approach to the asymptotic energy range. It also provides some control over the "effective coupling constant," leading to field theories with short- and long-distance behavior calculable in perturbation theory. We present below the result of a calculation of $\beta$.

The bare Lagrangian for a theory of interacting fermions and gauge mesons is

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}\{\partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu, B_\nu]\}^2 - \alpha^{-1}\,\mathrm{Tr}\{(\partial^\mu B_\mu)^2\} + 2\,\mathrm{Tr}\{\partial_\mu\varphi^*\partial^\mu\varphi + ig\partial_\mu\varphi^*[B_\mu,\varphi]\} + \bar\psi(i\not\partial - g\not B^a\sigma_a - M)\psi. \tag{1}$$

The gauge field is $B_\mu = \sum_a B_\mu{}^a\tau_a$; the $\tau_a$ are the matrices of the adjoint representation of $G$, a compact Lie group. The ghost field is $\varphi = \sum_a \varphi^a\tau_a$, the fermion field is $\psi = \sum_a\psi^a\sigma_a$, and the $\sigma_a$ are the matrices of $R$, the representation of $G$ under which the fermions transform. The term $\alpha^{-1}$ fixes the gauge. The group invariants which enter into the result are $C_2(G)$ and $C_2(R)$, the quadratic Casimir operators of the adjoint and fermion representations, and the trace $T(R)$ of the fermion representation, defined by $T(R)\delta_{ab} = \mathrm{Tr}\{\sigma_a\sigma_b\}$.

$\beta$ was calculated using the dimensional-regularization technique of 't Hooft and Veltman.[3] The scaling equations in this renormalization framework were derived by 't Hooft.[4] The Green's functions of the theory with $l$ fermions, $m$ gauge bosons, and $n$ ghosts satisfy the "new" renormalization-group equations[4-6]

$$\left[\mu\frac{\partial}{\partial\mu} + B(g)\frac{\partial}{\partial g} + \delta(g,\alpha)\frac{\partial}{\partial\alpha} + \hat\gamma_F(g)M\frac{\partial}{\partial M} + m\gamma_B(g,\alpha) + 2n\gamma_G(g,\alpha) + 2l\gamma_F(g,\alpha)\right]G^{(l,m,n)} = 0. \tag{2}$$

The renormalized coupling constant is $g$, the gauge parameter is $\alpha$, the renormalization scale parameter is $\mu$, and the fermion mass matrix is $M$. Within this renormalization framework, $\beta$ is independent of $\alpha$.[7] For simplicity, $\beta$ was evaluated in Feynman gauge ($\alpha = 1$).

The renormalization counterterms were fixed by requiring that they be of the form

$$Z = 1 + z/\epsilon + \text{higher order terms in } 1/\epsilon, \tag{3}$$

where $n$ is the dimension of space-time and $\epsilon = 4 - n$. $Z$ and $z$ are power series in $g^2$ and the gauge parameter $\alpha$. The counterterms calculated include those for the ghost propagator ($\tilde Z_3$), the ghost gauge boson vertex ($\tilde Z_1$), the gauge boson propagator ($Z_3$), and the triple boson vertex ($Z_1$). These are related by a Ward-Takahashi identity,[8] $Z_1/Z_3 = \tilde Z_1/\tilde Z_3$. To check the calculation, all four were computed independently. The connection between $\beta$ and the counterterms is given by[4,5]

$$\beta(g) = g^3(\partial/\partial g^2)(\tilde Z_1 - \tilde Z_3 - \tfrac{1}{2}Z_3) = g^3(\partial/\partial g^2)(Z_1 - \tfrac{3}{2}Z_3). \tag{4}$$

There are similar formulas for anomalous dimensions of the fields. The technique used to calculate the pole terms will be explained in detail in a future publication.

There are several internal checks on the calculation. No terms of the form $\ln q^2$ or (equivalently, in this theory) $\gamma_E$ (Euler's constant) should appear. The double-pole ($1/\epsilon^2$) terms may be independently calculated using the scaling equations of 't Hooft. Only the check using the Ward identities is nontrivial. Because of the complexity of the algebra, this check was considered necessary.

We find that

$$\beta(g) = -b_0 \frac{g^3}{16\pi^2} + b_1 \frac{g^5}{(16\pi^2)^2} + O(g^7), \quad b_0 = \tfrac{11}{3} C_2(G) - \tfrac{4}{3} T(R),$$
$$b_1 = -\tfrac{34}{3} C_2{}^2(G) + \tfrac{20}{3} C_2(G) T(R) + 4 C_2(R) T(R). \tag{5}$$

In the following examples we will specialize to SU($N$), and choose for $R$ the vector representation. We assume $m$ fermion multiplets. The relevant group invariants are then $C_2(G) = N$, $T(R) = \frac{1}{2}m$, and $C_2(R) = (N^2 - 1)/2N$.[1]

For SU(3), $\beta(g)/g$ near the origin is presented in Fig. 1. The effective perturbation-expansion parameter is $g^2 C_2(\mathrm{SU}(N))/16\pi^2$. We may choose the number of fermion multiplets to obtain the theory with the smallest value of the first nontrivial zero of $\beta$. With sixteen fermion multiplets we obtain a theory with calculable short-distance ($g \to 0$) and long-distance ($g \to g_1$) behavior, for $g$ initially between 0 and $g_1$. $g_1$ is $\beta$'s first zero past the origin, $g_1{}^2/16\pi^2 = 1/302 +$ higher-order corrections. The fractional correction due to the three-loop contribution is expected to be roughly $(g_1{}^2/16\pi^2)T(R) \sim \frac{1}{40}$, and is probably negligible. If the coefficient of the three-loop term is surprisingly large, we may consider SU($N$) for $N$ larger than 3. Choosing $m$ to minimize the first ($b_0$) term, we may ensure the existence of a zero of $\beta(g)$ valid in perturbation theory, even for abnormally large $g^7$ contributions.[9] $g_1$ is a simple zero of $\beta(g)$. As $g \to g_1$, the theory approaches a dilatation- and conformal-invariant Gell-Mann–Low limit theory.[10] Solving the renormalization group equations, with $\lambda$ scaling all momenta of the Green's functions ($t = \ln\lambda$), we find deviations from the limit theory smaller by powers of $\lambda$ rather than by logarithms (which characterize the $g \to 0$ end):

$$g_1{}^2 - \bar{g}^2(t,g) \underset{\lambda \to 0}{\simeq} \lambda^{2b_1(g_1{}^2/16\pi^2)^2} = \lambda^{1/453}, \tag{6}$$

for SU(3) with sixteen fermion multiplets.

For pure gauge theories the $g^5$ term is negative, and the approach to the asymptotic region is enhanced. We might expect the domain of attraction of the origin to be large (which would be appealing in a theory of the strong interactions). The effective coupling constant $\bar{g}(t,g)$ satisfies

$$d\bar{g}/dt = \beta(\bar{g}), \quad \bar{g}(0,g) = g. \tag{7}$$

As $\lambda \to \infty$ ($t \to \infty$), $g$ approaches zero as

$$\frac{\bar{g}^2(t,g)}{16\pi^2} \underset{t \to \infty}{\approx} \frac{1}{2b_0 t} + \frac{b_1}{4b_0{}^3}\frac{\ln t}{t^2} + O\left(\frac{1}{t^2}\right). \tag{8}$$

For SU(3) with three fermion triplets, which may be relevant for theories of the strong interactions,

$$\frac{\bar{g}^2(t,g)}{16\pi^2} \underset{t \to \infty}{\approx} \frac{1}{18t} - \frac{16}{729}\frac{\ln t}{t^2} + O\left(\frac{1}{t^2}\right). \tag{9}$$

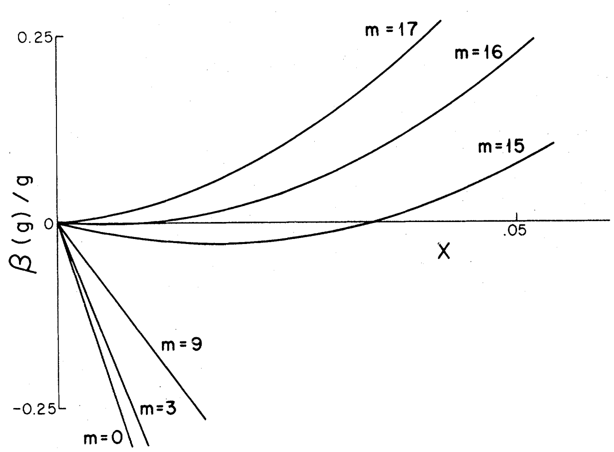


FIG. 1. Shape of $\beta(g)/g$ near the origin, for SU(3) with $m$ fermion multiplets. $x$ = effective expansion parameter = $(g^2/16\pi^2)C_2(\mathrm{SU}(3))$.

The second-order term slightly enhances the approach to zero. For $m=16$,

$$\frac{\bar{g}^2(t,g)}{16\pi^2} \underset{t\to\infty}{\approx} \frac{3}{2t}+\frac{1359}{2}\frac{\ln t}{t^2}+O\left(\frac{1}{t^2}\right), \tag{10}$$

and, as we expect for $\beta(g)$ small, $\bar{g}$ approaches its asymptotic values ($g=0$ and $g=g_1$) only slowly.

We do not expect to find a gauge theory of the above type where $\beta$ starts out positive and goes negative near enough to the origin for the zero to be valid in perturbation theory. If the lowest-order term is fixed to be zero, then

$$\beta(b_0=0)=g^5(16\pi^2)^{-2}[7C_2{}^2(G)+11C_2(G)C_2(R)]>0\,, \tag{11}$$

for an arbitrary gauge group $G$. This is approximately the case for SU(3) with $m=17$ ($b_0=-\frac{1}{3}$).

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[9]In theories as the above for which $g_1$ is calculable in perturbation theory, the effective coupling constant is constrained to remain small for all momenta. We expect a range of theories, for differing group structures, which have a finite zero of $\beta$. The possibility of dynamical quark confinement, arising from an infinite effective coupling constant in the infrared region, is not realized in this class of theories. Whether a group structure exists such that $\beta$ is negative definite, providing a possible mechanism for achieving "infrared slavery," is a nonperturbative question.

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# Is the Cross Section for $e^+e^- \to$ Hadrons Time Dependent in Colliding Beam Machines?*

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If $e^+ + e^- \to \Phi \to$ hadrons, where $\Phi$ is a scalar, radiative beam polarization in storage rings will induce an apparent time dependence of the cross section during an experimental run. This should provide a critical test for models which rely on such a scalar mechanism to enhance the cross section in the present energy range. A model of this kind is presented which identifies $\Phi$ with a composite Higgs field in gauge theories. We check for consistency with other processes and note that there may be an important relation to the shoulder in the dimuon mass distribution in $pp \to \mu^+\mu^- X$.

Synchrotron radiation in an $e^+e^-$ storage ring leads to transversely polarized beams where the electrons (positrons) are polarized antiparallel (parallel) to the guide magnetic field.[1] The magnitude of the polarization builds up toward a limiting value from the time of injection of the beams into the ring, with a characteristic time constant dependent upon the energy and other machine pa-