Self-Modulation and Self-Focusing of Electromagnetic Waves in Plasmas

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The nonlinear frequency shift of a strong electromagnetic wave in a plasma, due to weak relativistic effects and the $\vec{v}\times\vec{B}$ force, can cause modulation and self-focusing instabilities. These processes are explored, and their relation to self-focusing driven by the ponderomotive force is described.

We describe a new mechanism producing self-modulation and self-focusing of strong electromagnetic (EM) waves in plasmas. These effects occur because relativistic plasma dynamics produces a nonlinear index of refraction for the plasma. Even *weak* v^2/c^2 relativistic corrections can produce interesting self-modulation and -focusing, for parameters of high-power lasers presently existing or under construction.

Previous mechanisms for self-modulation and self-focusing of EM waves in collisionless plasmas[1-6] have used the ponderomotive force.[7] Joule heating can produce self-focusing in collisional plasmas.[5,8] The *temporal* evolution of these two processes relies on ion motion, and thus occurs with characteristic frequencies $|\omega^2| \lesssim \omega_{pi}^2 = 4\pi Ne^2/M_i$. The relativistic mechanism explored here requires motion only of the plasma electrons. Hence modulations of an incident wave can grow quickly compared to ion time scales, for incident light of sufficiently high intensity. Litvak[5] and Kidder[9] used nondynamical, geometrical optics considerations to estimate self-focusing lengths. Forslund, Kindel, and Lindman[10] described nonrelativistic modulational instability in a magnetic field. To our knowledge, the present analysis is the first treatment of EM modulational instability with $B_0 = 0$.

To derive the simplest features of relativistic self-modulation and self-focusing, we model the plasma as a cold uniform electron fluid with fixed ion density N, irradiated by a linearly polarized EM wave $\vec{E} = \hat{x}E_0\cos\chi_0$, $\chi_0 = \omega_0 t - k_0 z$. The usual relation between k_0 and ω_0 is modified, for two reasons.[11-13] First, because of their motion in the incident EM wave, electrons acquire a relativistic Lorentz factor $\gamma_0 \cong 1 + \frac{1}{2}(\nu_0 \sin\chi_0)^2$, where $\nu_0 \equiv eE_0/m_e c\omega_0 \ll 1$. This gives the electrons an amplitude-dependent mass in the plasma response, and results in an "inertial" current because of the difference between $\vec{p}$ and $\vec{v}$. Second, the $\vec{v}_0\times\vec{B}_0$ force on the electrons produces a density perturbation ΔN_0 having frequency $2\omega_0$: $\Delta N_0 = -N\nu_0^2(\cos 2\chi_0)(\omega_0^2 - \omega_p^2)/(4\omega_0^2 - \omega_p^2)$. The resulting nonlinear current $\Delta N_0 \vec{v}_0$ has a component at the pump frequency ω_0, and acts as a source for $\vec{E}_0$ and $\vec{B}_0$. The index of refraction due to these two nonlinear effects is[11]

$$n^2 = \frac{k_0^2 c^2}{\omega_0^2} = 1 - \frac{\omega_p^2}{\omega_0^2} + \frac{\nu_0^2\omega_p^2}{2\omega_0^2}\left(\frac{3}{4} - \frac{\omega_0^2 - \omega_p^2}{4\omega_0^2 - \omega_p^2}\right), \qquad (1)$$

where $\omega_p^2 \equiv 4\pi Ne^2/m_e$.[14]

To study the *dynamics* of self-modulation and self-focusing, we use a linear instability analysis. Our equilibrium is the EM wave described above, together with its second-harmonic density perturbation, ΔN_0, and the nonlinear $\omega_0(k_0)$ given by (1). Linearized quantities vary in the y-z plane. The perturbed vector potential $\delta\alpha_x \equiv e\delta A_x/m_e c^2$ (Coulomb gauge) obeys the linearized wave equation

$$\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\delta\alpha_x = \omega_p^2\left[\frac{\delta v_x}{c}\left(\frac{1+\Delta N_0}{N}\right) + \frac{\delta n}{N}\frac{v_{0x}}{c}\right], \qquad (2)$$

where δn is the first-order electron density perturbation, and $v_{0x}/c = -\nu_0 \sin\chi_0 + O(\nu_0^3)$.

To find a self-consistent solution, we must know the values of δv_x and δn. In terms of the dimensionless momentum $u_x = p_x/m_e c$, δv_x is given by

$$\delta v_x/c = (\delta u_x/\gamma_0) - \delta\gamma(u_{x0}/\gamma_0^2) = (\delta u_x/\gamma_0^3). \qquad (3)$$

Conservation of x canonical momentum relates δu_x with the vector potential: $\delta u_x = \delta\alpha_x \equiv e\delta A_x/$

$m_e c^2$. Hence Eq. (3) becomes[15]

$$\delta v_x/c = \delta\alpha_x[1 - \tfrac{3}{2}\nu_0^2 \sin^2\chi_0] + O(\nu_0^3). \quad (4)$$

The next task is to find the electron density perturbation δn. It is sufficient to have the wave equation correct to order $\nu_0^2 \sim |v_{0x}^2|/c^2$. Since δn appears in Eq. (2) multiplied by v_{0x}, we need only find δn correct to order ν_0. The continuity equation and Lorentz-force equation are

$$\partial^2\delta n/\partial t^2 + N\nabla\cdot(\partial\vec{\delta v}/\partial t) = 0 + O(\nu_0^2), \quad (5)$$

$$\partial\delta\vec{v}/\partial t = e\nabla\delta\varphi/m_e - \hat{z}k_0c^2\nu_0\cos\chi_0\delta\alpha_x + c^2\nu_0\sin\chi_0\nabla\delta\alpha_x + O(\nu_0^2), \quad (6)$$

where $\delta\varphi$ is the electrostatic potential perturbation.

We now put the pieces together and derive a dispersion relation. Take the Fourier transform of Eqs. (2) and (4)–(6) in y, z, and t. Write (2) as two equations for the two quantities $\delta\alpha_x^\pm \equiv \delta\alpha_x(\vec{k}\pm\vec{k}_0, \omega\pm\omega_0)$. Substitute the appropriate Fourier amplitudes of δn and δv_x, determined from (4)–(6), into the right-hand side of these two equations for $\delta\alpha_x^\pm$. The equations for $\delta\alpha_x^\pm$ involve Fourier components of the electron density at $\delta n^{\pm 2}$ and at δn^0. For ponderomotive force mechanisms of self-modulation and self-focusing, it is the δn^0 component which dominates. In contrast, for the present relativistic process the $\delta n^{\pm 2}$ terms are more important in the ratio $\omega_p^2/k^2c^2 \gg 1$. For the instabilities we discuss here, the two scattered EM waves $\delta\alpha_x^\pm$, and their accompanying density perturbations $\delta n^{\pm 2}$, closely resemble the *driving* wave $E_0\cos\chi_0$ and its second-harmonic density perturbation ΔN_0. The components $\delta\alpha_x^{\pm 3}$ contribute terms smaller than $O(\nu_0^2)$, and hence will be neglected.

The above procedures result in two homogeneous equations for the Fourier amplitudes $\delta\alpha_x^\pm$. The dispersion relation is the requirement that the determinant of the coefficients vanish. We assume $|\omega| \ll (\omega_p^2/\omega_0)$ and $k^2 \ll k_0^2$ [ω and $\vec{k}$ correspond to the low-frequency density perturbation $\delta n^0 \equiv \delta n(\omega, \vec{k})$], and obtain

$$(\omega^2 - k^2c^2)^2 - 4(\omega\omega_0 - \vec{k}\cdot\vec{k}_0c^2)^2 + \nu_0^2\omega_p^2 q(\omega^2 - k^2c^2) = 0, \quad (7)$$

$$q \equiv \tfrac{3}{4} - (\omega_0^2 - \omega_p^2)/(4\omega_0^2 - \omega_p^2). \quad (8)$$

To solve Eq. (7) for self-modulation, we take $\vec{k}\,\|\,\vec{k}_0$ and set $\omega = k(c^2k_0/\omega_0) + \Delta\omega \equiv kV_g^0 + \Delta\omega$. With the ordering $\Delta\omega \sim \nu_0^2\omega_0$, $k \sim \nu_0(\omega_0/c)$, we retain terms through $O(\nu_0^4)$ in (7) and find

$$\Delta\omega \cong (kc/2)(\omega_p/\omega_0)^2[(kc/\omega_0)^2 - \nu_0^2 q]^{1/2}. \quad (9)$$

The first term in the square brackets represents the mismatch between the sidebands and the pump. For simplicity of presentation we have also taken $\omega_0^2 \gg \omega_p^2$. There is instability for $k < (\omega_0/c)\nu_0 q^{1/2}$. The maximum growth rate occurs for $k = (\omega_0/c)\nu_0(q/2)^{1/2}$, and is given by

$$\gamma_{\max} = (\omega_p^2/4\omega_0)\nu_0^2 q. \quad (9a)$$

The maximum spatial growth rate is $\kappa_{\max} = \gamma_{\max}/V_g^0$. During the instability, the driving wave acquires two growing EM sidebands, at frequencies $\omega_0 \pm \vec{k}\cdot\vec{V}_g^0$ and wave numbers $(k_0 \pm k)\hat{z}$. The presence of these sidebands is equivalent to modulation of the driving wave, because the *total* EM response is of the form

$$\sin\chi_0 + \epsilon[\sin(\chi_0+\varphi) + \sin(\chi_0-\varphi)] \cong \sin(\chi_0 + 2\epsilon\sin\varphi), \quad \epsilon \ll 1.$$

A simple argument leads to possible modulational instability for *any* weakly nonlinear wave with dispersion relation $\omega(k,\nu) = \omega_0(k) - A\nu^2$, with ν the wave amplitude.[16] From the phase function $\theta(z,t)$ we define $k \equiv \partial\theta/\partial z$, $\omega \equiv -\partial\theta/\partial t$. Then the relation $\partial k/\partial t = -\partial\omega/\partial z$ requires

$$(\partial/\partial t + V_g^0\partial/\partial z)k = A\partial\nu^2/\partial z. \quad (10)$$

Consider a sinusoidal perturbation $\Delta\nu$ to the original wave amplitude ν_0 (Fig. 1). In a frame moving with the group velocity $V_g^0 \equiv (\partial\omega_0/\partial k)_{k=k_0}$, Eq. (10) says that the wave number $\Delta k = k - k_0$ changes in time at the rate $2\nu_0 A\partial\Delta\nu/\partial z_0$. Here z_0 is a Lagrangian coordinate specifying position on the wave envelope, and A is positive. In region *a* of

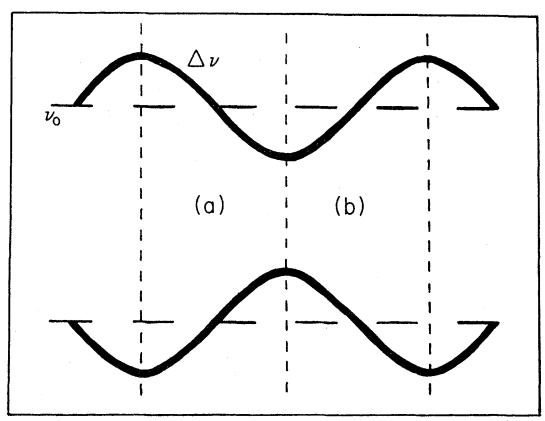

FIG. 1. The original wave envelope ν_0, and the modulated envelope $\nu_0 + \Delta\nu$, during self-modulation.

Fig. 1, where $\partial\Delta\nu/\partial z_0<0$, Δk decreases with time. When $\partial V_g^0/\partial k$ is positive, the decrease of Δk implies that V_g decreases with time. Thus in region a, energy slowly moves backward in the frame moving with the average group velocity V_g^0. Analogous arguments imply that energy slowly propagates forward in region b. Thus energy accumulates at the local maxima of $\Delta\nu$, causing a purely growing instability in the frame moving with V_g^0. The ponderomotive force is not a major effect in the dynamics of the instability.

For the contrasting case of self-focusing we take $\vec{k}\cdot\vec{k}_0=0$, and assume $\omega\sim\nu_0^2\omega_0$, $k\sim\nu_0(\omega_0/c)$. We retain terms through $O(\nu_0^4)$ in (7) to obtain

$$\omega\cong\pm\tfrac{1}{2}ikc[(kc/\omega_0)^2-\nu_0^2q(\omega_p/\omega_0)^2]^{1/2}. \tag{11}$$

The first term in square brackets represents diffraction, and is stabilizing. The maximum wave number for instability is given by $k<(\omega_p/c)\nu_0q^{1/2}$, smaller by the factor ω_p/ω_0 than the analogous k for self-modulation. The maximum growth rate is the *same* as for self-modulation, but occurs at a k smaller by the factor ω_p/ω_0. Computer simulations[17] support the growth rates predicted by Eqs. (9) and (11). Steady-state self-focusing can be understood by arguments from geometrical optics. By Eq. (1), if the light intensity ν_0^2 is locally enhanced at a point, the phase velocity is decreased there relative to its value on either side. The wave front curves, further enhancing the intensity by focusing the light "downstream" in the $\hat{k}_0$ direction.

For simplicity we have neglected ion motion, and hence have suppressed ponderomotive-force–driven instabilities which compete with the processes considered here. We emphasize that, for high laser intensities, the relativistic instabilities derived here can grow so quickly in time that ion inertia prevents the ions from following along. For self-focusing we estimate the intensities when this can occur, by comparing the growth rate due to ponderomotive force, $\gamma_{\rm pond}\cong\nu_0\omega_{pi}/2^{1/2}$, with the maximum relativistic growth rate (9a). The relativistic process is more important than the ponderomotive force, for the *temporal* development of self-focusing, if $\nu_0>(\omega_0\omega_{pi}/\omega_{pe}^2)(2^{1/2}/q)$. A more detailed study of these two mechanisms for self-focusing verifies this simple estimate.[17] Neglect of ion motion in the modulational instability of Eq. (9) is justified when $|\omega|=kV_g^0>\omega_{pi}$. For the k of maximum growth, this condition is $\nu_0>(\omega_{pi}/ck_0)(2/q)^{1/2}$.

The assumptions $|\omega|\ll\omega_p^2/\omega_0$, $k^2\ll k_0^2$, and $k^2\ll\omega_p^2/c^2$ necessary for the above results to hold are summarized by two inequalities. For an underdense plasma we must have $\nu_0\ll(\omega_p/\omega_0)^2$. Near critical density, on the other hand, the condition $k_0c\gg\nu_0\omega_0$ must be satisfied.

The temporal growth rates represented by Eqs. (9) and (11) are interesting for high-power lasers. For example, consider a Nd-glass laser ($c/\omega_0=1.06$ μm), $I=5\times10^{16}$ W/cm^2, incident on a plasma with $\omega_0=1.7\omega_{pe}$. Then the fastest growth time is 6×10^{-13} sec for both modulational and self-focusing instabilities. The exponentiation length for these processes is 140 μm. For self-modulation the modulating wavelength is 14 μm, and the modulating frequency a tenth of the laser frequency. For self-focusing, the filament width for fastest growth is 23 μm. Intensities of $\sim10^{16}$ W/cm^2 are probably higher than those contemplated for laser fusion applications. However, such lasers are presently coming into use in target experiments.[18] The above numbers indicate that modulations and filaments may exponentiate significantly if the strong laser pulse lasts longer than a few picoseconds, or if the blow-off plasma becomes larger than a few hundred microns. Modulations could represent a mechanism to *broaden the frequency spectrum* of the incident laser. This effect may be important because resonant instabilities (e.g., stimulated Brillouin scattering or the parametric decay instability) can be inhibited if the driving wave is not monochromatic.

A major unanswered question about relativistic self-modulation and focusing concerns competition between these processes and other laser-plasma instabilities.[1,4,10] The Raman process has a far larger growth rate, and if it occurs it should be dominant for $\omega_0>2\omega_{pe}$. Even for $\omega_0<2\omega_{pe}$, strong-coupling Brillouin scattering has a larger growth rate than the relativistic instabilities considered here. However, density gradients and frequency broadening can stabilize Raman and Brillouin scattering. At present we see no mechanism for density-gradient stabilization of relativistic self-modulation and focusing, and we argue that self-modulation may itself cause frequency broadening. The presence or absence of these processes may ultimately depend on "nonideal" effects, such as spatial nonuniformity and frequency spread of the incident laser beam.

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Quasiparticle Lifetimes and Microwave Response in Nonequilibrium Superconductors*

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A theoretical treatment is presented of the quasiparticle-recombination dynamics of a nonequilibrium superconductor. The microwave reflectivity of a superconducting film in the presence of external pair breaking is calculated. It is shown how microwave experiments may permit the separation of the effective recombination time into its intrinsic and recombination-phonon components.

Testardi[1] has shown experimentally that when a thin superconducting film is driven normal by intense pulsed optical radiation, the transition displays characteristics which cannot be accounted for by simple lattice heating. He suggested that excess quasiparticles created by photon-induced pair breaking were responsible. Motivated by this observation, Owen and Scalapino[2] investigated a modified BCS model of a superconductor in which the quasiparticle density is maintained at a level above the thermal-equilibrium value by an external source of dynamic pair breaking. Among the predictions of this model are a dependence of the energy gap on the excess quasiparticle density, and a first-order transition to the normal state at a critical excess quasiparticle density. Some of the model's predictions have been experimentally confirmed by Parker and Williams[3] using tunnel junctions irradiated with laser light.

Some time ago, Rothwarf and Taylor[4] pointed out the crucial importance of recombination phonons in the coupled quasiparticle-pair–phonon system in a nonequilibrium superconductor. The interpretation of nonequilibrium experiments using tunnel junctions depends on assumptions about the behavior of these phonons in the relatively complex junction structure. It would obviously be desirable to test the Owen-Scalapino model in a simple thin film, in which the phonons can be accounted for with greater certainty. The use of microwave-frequency reflection, absorption, or transmission measurements as a probe of the quasiparticle density makes this possible. The quasiparticle density can be monitored in the superconducting state in contrast to the quasi-dc experiment of Testardi, in which only the transition to the normal state could be