lator wave functions is not crucial to the result. A more detailed treatment of this color-symmetry breaking effect, including the case when the symmetry breaking is relatively small, will be published elsewhere.

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[1]P. J. Litchfield, in Rapporteur's Talk at Proceedings of the Seventeenth International Conference on High Energy Physics, London, England, 2-10 July 1974 (to be published).

[2]G. Karl and E. Obryk, Nucl. Phys. B8, 609 (1968).

[3]D. B. Lichtenberg, Phys. Rev. 178, 2197 (1968).

[4]S. Ono, Prog. Theor. Phys. 48, 964 (1972).

[5]If color symmetry is exact, this mechanism leads to color singlets as the lightest states as shown by Y. Nambu, in *Preludes in Theoretical Physics,* edited by A. de-Shalit, H. Feshbach, and L. Van Hove (North-Holland, Amsterdam, 1966), p. 133.

Infrared Behavior of Yang-Mills Theories

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The ultraviolet freedom of non-Abelian gauge theories is used to suggest the presence of a new symmetry of the exact theory which is not present in finite orders of perturbation theory. The infrared behavior of these theories is determined as a consequence of this symmetry.

There is a distinct possibility that the strong interactions may be described by an unbroken non-Abelian (generalized Yang-Mills[1]) gauge theory.[2] The gauge group is presumably colored SU(3) with the gauge fields A_μ^a in the adjoint representation ($a=1-8$) and the quark fields ψ in the (triplet) fundamental representation. The quark fields should also be ordinary SU(3) triplets or simple (charmed) generalizations. Such a theory is asymptotically free and so has a computable ultraviolet behavior.[3] More importantly, the theory is therefore *not* infrared free and so offers a unique possibility of providing for color confinement.[2]

Because the theory is not infrared free, it has not been possible to perform any reliable calculations which could determine what actually happens in the infrared region.[4] The hoped for confinement cannot be investigated with conventional perturbation or renormalization-group[5] techniques.[6] In this note, we shall attempt to overcome this impasse and argue that, precisely because of the lack of infrared freedom (or the corresponding presence of ultraviolet freedom), exact statements *can* be made about the infrared behavior of the theory. Unlike conventional zero-momentum theorems, our statements, which are consequences of renormalization, should be valid only in the exact theory and not classically or in finite orders of perturbation theory. Our present results do not answer the confinement question, but, because they constitute hopefully exact statements about the infrared behavior of non-infrared-free theories, they should provide new tools for investigating this and related problems.

Our approach is as follows. We consider the renormalized gauge field equations in the form

$$\partial^\nu[\partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x)] = J_\mu^a(x), \tag{1}$$

where $J_\mu^a(x)$ contains the usual Yang-Mills self-couplings and couplings to quark fields and ghost fields, gauge-fixing terms (we work in the generalized-Lorentz-gauge formalism), and all necessary counterterms.[7] Classically, or order by order in perturbation theory, (1) possesses no interesting symmetry besides the usual Poincaré and non-Abelian gauge invariance. Asymptotic freedom, however, enables us to make precise statements about the exact source operator $J_\mu^a(x)$ obtained by summing the perturbation series.[8] We will argue that this exact source is invariant under the transformation (R transformation)

$$A_\mu^a(x) \to A_\mu^a(x) + r_\mu^a, \tag{2}$$

for arbitrary constants r_μ^a, $a=1-8$, $\mu=0-3$. The exact field equations (1) are therefore invariant to (2) and so (2) is an exact (spontaneously broken) symmetry of the exact theory. Our zero-momentum theorems are the Ward-Takahashi (WT) identities appropriate to this symmetry.

Before outlining the derivation of these results,[9] we will illustrate them in the simpler context of conventional quantum electrodynamics (QED).[10] We work in the Gupta-Bleuler Lorentz-gauge formalism. The invariance of the theory under the gauge transformation[11]

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{3}$$

implies the conventional WT identities, the simplest of which expresses the transversality of the vacuum polarization tensor $\Pi_{\mu\nu}(q)$:

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2). \tag{4}$$

This implies that the transverse part of the photon propagator

$$D_{\mu\nu}(q) \equiv -(2\pi)^{-4}\int d^4x\, e^{iqx}\langle 0|TA_\mu(x)A_\nu(0)|0\rangle \tag{5}$$

has the form

$$D(q^2) = [q^2 + q^2\Pi(q^2)]^{-1}, \tag{6}$$

but says nothing about the behavior of (6) for $q^2 \to 0$. In perturbation theory, one maintains that $\Pi(0)=0$ so that (6) has the photon pole at $q^2=0$, but a singularity in $\Pi(q^2)$ at $q^2=0$ would lessen the singularity in (6), and a pole at $q^2=0$,[12]

$$\Pi(q^2) \sim -M^2/q^2, \tag{7}$$

would completely remove the singularity. Gauge invariance alone thus does not say anything about $D(0)$ and so the physical photon cannot be considered as a Goldstone boson arising from the spontaneous breakdown of gauge invariance.

Suppose now that the gauge group G of QED contained the special class of gauge functions

$$\Lambda(x) = R(x) \equiv r^\mu x_\mu, \quad r_\mu = \text{const}, \tag{8}$$

under which

$$A_\mu(x) \to A_\mu(x) + r_\mu. \tag{9}$$

A simple way to determine the implications of this is to consider the product

$$\Pi_\mu{}^\kappa(q)D_{\kappa\nu}(q) \equiv e(2\pi)^{-4}\int d^4x\, e^{iqx}\langle 0|TJ_\mu(x)A_\nu(0)|0\rangle, \tag{10}$$

where $J_\mu(x)$ is the conserved electric current operator. The invariance of (10) to (3) for square-integrable $\Lambda(x)$, together with the fact that (5) acquires the term $q_\mu\hat{\Lambda}(q)\Lambda_\nu(0)$[13] under (3), implies that $q^\mu\Pi_{\mu\nu}(q)=0$ and hence (4). The invariance of (10) to (9), together with the fact that (5) acquires the term $r_\mu r_\nu \delta^4(q)$ under (9), implies that $\Pi_{\mu\nu}(q)\delta^4(q)=0$ and hence that $\Pi(q^2)$ is less singular than (7) at $q^2=0$, so that (6) must have a singularity at $q^2=0$.[14] It is therefore not the existence of a gauge-invariance group G but the presence of (8) in G which implies the presence of a physical zero-mass excitation. In conventional four-dimensional QED, $R \in G$[10] and so the physical photon can be interpreted as the Goldstone boson arising from the spontaneous breakdown of R invariance. In exactly soluble[12] two-dimensional massless QED, $R \notin G$[10] and so there the photon can (and does) acquire a mass. The further consequences of R invariance can best be deduced and stated by functional methods. The general WT identity is

$$\int d^4x\left[\frac{\delta\Gamma}{\delta\mathfrak{A}_\mu(x)} + ex_\mu\frac{\delta\Gamma}{\delta\Psi(x)}\Psi(x) - ex_\mu\frac{\delta\Gamma}{\delta\bar{\Psi}(x)}\bar{\Psi}(x)\right] = 0, \tag{11}$$

where $\Gamma(\mathfrak{A}_\nu, \Psi, \bar{\Psi})$ is the generating functional of proper vertices. In particular, the proper n-photon amplitude vanishes whenever any external momentum vanishes.

We return now to the non-Abelian gauge theory (1). In terms of the renormalized fields A_μ^a, ψ, C_1, and C_2 (the C_i are the ghost fields), the renormalized coupling constant g, the renormalized gauge parameter α, and the usual renormalization constants Z_i, the formal expression for the source operator is[7,15]

$$-J_\mu{}^a=(Z_1/Z_3)gf^{abc}\{A^{c\nu}[\partial_\mu A_\nu{}^b-\partial_\nu A_\mu{}^b+(Z_1/Z_3)gf^{bde}A_\mu{}^dA_\nu{}^e]+\partial^\nu(A_\mu{}^bA_\nu{}^c)\}$$
$$+(1/Z_3\alpha)\partial_\mu\partial_\nu A^{a\nu}+(Z_1/Z_3{}^2)g(Z_2\bar\psi\gamma_\mu T^a\psi+\tilde{Z}_3f^{abc}\partial_\mu C_1{}^bC_2{}^c). \qquad (12)$$

The renormalization constants Z are functions of a suitable cutoff parameter K and their behavior for large K [all that is relevant in (12)] can be exactly computed in asymptotically free theories by using renormalization-group techniques.[16] The result is that the ratio Z_1/Z_3 vanishes in all asymptotically free theories for any choice of α[9]:

$$\lim_{K\to\infty}[Z_1(K)/Z_3(K)]=0. \qquad (13)$$

(In finite orders of perturbation theory, on the contrary, Z_1/Z_3 is logarithmically divergent.) When the R transformation (2) is performed in (12), there results additional terms only of the forms

$$(Z_1/Z_3)r^\nu\partial_\mu A_\nu, \quad (Z_1/Z_3)^2r^\nu A_\mu A_\nu. \qquad (14)$$

Since $\partial_\mu A_\nu$ and $(Z_1/Z_3){:}A^2{:}$ are finite field operators, the products (14) vanish because of (13). The operator (12) is therefore formally R invariant and hence so is the field equation (1). The same is true for the quark and ghost field equations.

At a less formal level, the source operator (12) can be expressed as the local limit of a nonlocal field product[17]:

$$J_\mu{}^a(x)=\lim_{\xi\to0}[\ldots+g(1+bg^2\ln\xi^2\mu^2)^{-\gamma}f^{abc}A^{c\nu}(x+\xi)\partial_\mu A_\nu{}^b(x)+\ldots], \qquad (15)$$

where we have exhibited only a typical term. The positive constants b and γ are exactly computable via the renormalization group.[9] μ^2 is the Euclidean subtraction point. The R invariance of the exact expression (15) is evident, as is the R noninvariance of the expression in finite orders of g.[18]

Before stating the consequences of these observations, we should emphasize the nonrigorous nature of our argument for R invariance. We have freely interchanged the R transformation (2) and the regularization removal limit,[19] and we have not taken account of the need to employ a gauge-invariant regularization before making the renormalization subtractions indicated in (12).[20] We unfortunately cannot check that our procedure is legitimate in perturbation theory since our conclusions should only be true for the exact theory.[21] In this sense, the proposed R invariance of Yang-Mills theories is on a very different footing from the R invariance of QED discussed above. Note also that all our remarks refer to the theory obtained by summing the perturbation series and not to other possible solutions of the field equations. These reservations should be kept in mind below.

The consequences of R invariance of the theories under consideration can be summarized by the WT identity

$$\int d^4x\,[\delta\Gamma(\mathfrak{a},\ldots)/\delta\mathfrak{a}_\mu{}^a(x)]=0, \qquad (16)$$

where $\Gamma(\mathfrak{a},\ldots)$ is the generating functional of proper vertices and $\mathfrak{a}_\mu{}^a(x)$ is a classical Yang-Mills field. This implies the vanishing of the proper n-boson amplitude whenever an external momentum vanishes:

$$\Gamma^{a_1\cdots a_n}_{\alpha_1\cdots\alpha_n}(q_1,\ldots,q_n)|_{q_i=0}=0. \qquad (17)$$

In particular, the inverse photon propagator vanishes at $q=0$,

$$(D^{-1})_{\mu\nu}{}^{ab}(0)=0, \qquad (18)$$

and this rules out the Schwinger mechanism (7) in the generalized Lorentz gauges. The zero-momentum behaviors (17) suggest that the infrared behavior is not as bad as is indicated in finite orders of perturbation theory. Our analysis does not specify the rate of vanishing in (17), but the form of (15), as well as model calculations,[21] suggests a logarithmic vanishing. Note finally that the results (17) do not directly bear on the existence of an S matrix. Progress in this direction would require information about the limits $q_i{}^2\to0$ without $q_i\to0$.

We find it most interesting that the exact infrared statement (17) can be (admittedly nonrigorously) deduced in asymptotically free quantum field theories. We recall the indirect nature of our derivation: The *ultraviolet* freedom determined the exact form of the field equation (1), as in (15), and this form was invariant to the R transformation (2), which implied the *infrared* behaviors (17) as WT identities. Similar methods can be used to deduce exact theorems at all momenta.[9] This leads us to hope that a complete solution to the infrared and confinement problem in these physically interesting theories may be attainable.

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[1]C. N. Yang and R. Mills, Phys. Rev. 96, 191 (1954).

[2]H. Fritzsch, M. Gell-Mann, and H. Leutwyler, Phys. Lett. 47B, 365 (1973); S. Weinberg, Phys. Rev. Lett. 31, 494 (1973).

[3]G. t'Hooft, unpublished; H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. Gross and F. Wilczek, *ibid.*, 1343 (1973).

[4]This is in contrast to QED and φ^4 theory where the essential infrared freedom enables the infrared problem to be solved. See K. Symanzik, Commun. Math. Phys. 34, 7 (1973).

[5]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); K. Wilson, Phys. Rev. D 3, 1818 (1970); C. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[6]Various possibilities are considered by T. Appelquist, to be published.

[7]For a comprehensive review, see E. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1974).

[8]In this paper, an "exact" quantity will mean a quantity obtained by summing the entire perturbation series, assumed to be at least an asymptotic expansion.

[9]A complete discussion is given in R. A. Brandt and Ng W.-C., DESY Reports No. 74/37 and No. 74/38 (to be published).

[10]A complete discussion is given by R. A. Brandt and Ng W.-C., Phys. Rev. D 10, 1918 (1974).

[11]The spinor fields must also be transformed in the usual way.

[12]J. Schwinger, Phys. Rev. 128, 2425 (1962).

[13]$\hat{\Lambda}(q) \equiv \int dx\, e^{iq\cdot x}\Lambda(x)$. $\Lambda_\nu(0) \equiv \partial_\nu \Lambda(x)|_{x=0}$.

[14]This conclusion depends on the R invariance of the theory expressed in terms of independent fields. See Ref. 10.

[15]Vacuum subtractions are suppressed. A color-symmetric fermion mass term can be included without changing our conclusions.

[16]Ng W.-C. and K. Young, Phys. Lett. 51B, 291 (1974); T. P. Cheng, Ng W.-C., and K. Young, Phys. Rev. D 10, 2459 (1974).

[17]The corresponding expression for the electric current in QED is given in R. A. Brandt, Ann. Phys. (New York) 52, 122 (1969).

[18]For a previous related use of R invariance, see R. A. Brandt and Ng W.-C., Nuovo Cimento 13A, 1025 (1973), and Phys. Rev. D 9, 373 (1974).

[19]This works in QED. See Ref. 17.

[20]This means that further counterterms may be effectively present in (12). We must assume that such terms do not violate R invariance. Note that a potentially dangerous mass counterterm would ruin renormalizability and so cannot be present.

[21]Partial diagrammatic summations which support our conclusions are given in Ref. 9.