Analytical Calculation to All Orders in $Z\alpha$ of Vacuum Polarization at Short Distances*

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We have calculated the leading terms of the potential due to vacuum polarization about a point nucleus, $V(r)=m[a_{-1}(mr)^{-1}+a_0+a_1(mr)+a_{2\lambda}(mr)^{2\lambda}]$. Our results for a_1 and $a_{2\lambda}$, to all orders in $Z\alpha$, extend the previously known results which included terms only up to $(Z\alpha)^3$. For a_{-1} we confirm previous results. The new portions of a_1 and $a_{2\lambda}$ do not remove present discrepancies in muonic x-ray transition such as $5g\rightarrow 4f$ in ^{208}Pb.

Quantum electrodynamics predicts deviations from pure Coulombic behavior near a point charge. The potential energy of a negative charge $-e$ located a distance r from a point charge Ze is not given entirely by the Coulomb expression, $V(r)=-Z\alpha/r$ (we set $\hbar=c=1$). There is a correction of order $\alpha Z\alpha$, the Uehling potential.[1,2] It arises from the polarization of the vacuum of virtual electrons of mass m and is given by

$$V_{\text{Ueh}}(r)=-\frac{\alpha Z\alpha}{\pi r}\int_{2m}^{\infty}d\kappa\, e^{-\kappa r}\left(\frac{2}{3\kappa^2}+\frac{4m^2}{3\kappa^4}\right)(\kappa^2-4m^2)^{1/2} \tag{1}$$

$$\simeq\frac{\alpha Z\alpha}{\pi r}\left[\tfrac{2}{3}(\ln mr+\gamma)+\tfrac{5}{9}-\tfrac{1}{2}\pi(mr)+(mr)^2-\tfrac{2}{9}\pi(mr)^3-\tfrac{1}{6}(mr)^3(\ln mr+\gamma)+O(mr)^4\right], \tag{2}$$

where $\gamma\cong 0.577\,21$ is Euler's constant.

The vacuum-polarization charge density to all orders in $Z\alpha$ is given by a trace of the Green's function for an electron in a Coulomb field, evaluated at two identical space-time points. The sole divergence that occurs is an infinite charge renormalization in order $\alpha Z\alpha$: The charge density away from the origin is finite in all orders in $Z\alpha$. In the only previous calculation done to all orders in $Z\alpha$, Wichmann and Kroll[1] found that the vacuum-polarization charge density included a point charge, $\delta Q'\delta^3(\vec{r})$, where $\delta Q'$ is a function of $Z\alpha$ which they expressed as a double infinite sum. Proceeding from expressions of Wichmann and Kroll, the potential of order $\alpha(Z\alpha)^3$ has previously been obtained for the small-distance domain[2,3]:

$$V_{13}(r)=\frac{\alpha(Z\alpha)^3}{\pi r}\left\{\left[-\tfrac{2}{3}\zeta(3)+\tfrac{1}{6}\pi^2-\tfrac{7}{9}\right]+\left[2\pi\zeta(3)-\tfrac{1}{4}\pi^3\right]mr+\left[-6\zeta(3)+\tfrac{1}{16}\pi^4+\tfrac{1}{6}\pi^2\right](mr)^2 +\left[\tfrac{2}{9}\pi\gamma+\tfrac{2}{3}\pi\zeta(3)+\tfrac{2}{9}\pi\ln 4mr-\tfrac{31}{27}\pi\right](mr)^3+O((mr)^4)\right\}. \tag{3}$$

The work we report here is based on techniques not heretofore applied to this problem. The first involves Fredholm determinants and simplifies the calculation of the induced point charge and the other terms in the potential with integral powers of mr. The second is the use of Mellin-Barnes representations of the confluent hypergeometric functions which occur in the Coulomb-Dirac Green's function. This representation provides a unified approach to the expansion in mr and simplifies especially the calculation of the terms with nonintegral powers of mr. The details of our calculations will be given elsewhere.[4]

Our method shows the form of the full expansion of the vacuum polarization potential in ascending powers of mr to be

$$V(r)=m\left[\sum_{l=0}^{\infty}a_{2l-1}(mr)^{2l-1}+a_0+\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}a_{2\lambda_k+n}(mr)^{2\lambda_k+n}\right], \tag{4}$$

where the a's are functions of $Z\alpha$, and in the second sum $\lambda_k=[k^2-(Z\alpha)^2]^{1/2}$. Although all the coefficients (save a_0) can be calculated by our procedures, the terms a_{-1}, a_1, and $a_{2\lambda}$ are adequate to evaluate accurately the energy shifts in heavy muonic atoms. The a_0 term does not affect any transition energy.

Our result for the induced point charge is

$$\delta Q'=\sum_{k=1}^{\infty}(4ek/\pi)\operatorname{Im}\left[(\lambda_k-iZ\alpha)\psi(\lambda_k-iZ\alpha)-\ln\Gamma(\lambda_k-iZ\alpha)-\tfrac{1}{2}\ln(\lambda_k-iZ\alpha)+iZ\alpha k\psi'(k)-iZ\alpha/2k\right], \tag{5}$$

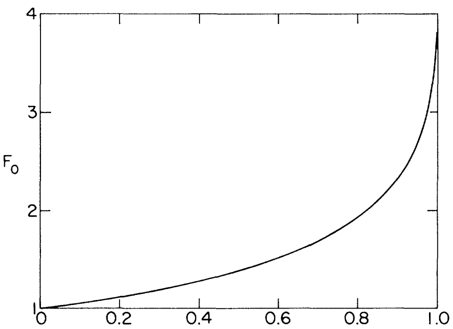

FIG. 1. F_0 as a function of $(Z\alpha)^2$. F_0 represents the correction to the $(Z\alpha)^5$ approximation to the induced point charge, Eq. (5).

TABLE I. The energy shifts of the $5g_{9/2}$ and $4f_{7/2}$ levels in muonic ^{208}Pb, in eV. Our new results, Eqs. (12) and (13), replace the previous approximations, Eqs. (14) and (15). All Uehling order terms are removed according to Eqs. (12)–(15).

Equation	$5g_{9/2}$	$4f_{7/2}$	$E_5 - E_4$
(12)	49.2	32.2	17.0
(13)	−23.0	−13.9	−9.1
(12) + (13)	26.2	18.3	7.9
(14)	29.6	19.4	10.2
(15)	−9.3	−5.7	−3.6
(14) + (15)	20.3	13.7	6.6
Modification:			
(12) + (13) − (14) − (15)	5.9	4.6	1.3

where $\psi(z) = d\ln\Gamma(z)/dz$. This agrees with the result of Wichmann and Kroll[1] although they obtained a rather differently appearing expression.

Following Wichmann and Kroll, we display $\delta Q'$ by expanding in powers of $Z\alpha$:

$$\delta Q' = (e/3\pi)\{(Z\alpha)^3[2\zeta(3) + \tfrac{7}{3} - \tfrac{1}{2}\pi^2] - (Z\alpha)^5[2\zeta(5) + \tfrac{71}{5}\zeta(3) - 47\pi^4/240] + O((Z\alpha)^7)\}$$
$$= -e[(Z\alpha)^3(0.020\,940) + (Z\alpha)^5(0.007\,121)F_0(Z\alpha)], \tag{6}$$

where $F_0(0) = 1$. Our numerical results for F_0, shown in Fig. 1, are in approximate agreement with those of Wichmann and Kroll.[1]

For the term in the potential of order mr we find

$$a_1 = \alpha(Z\alpha)\sum_{k=1}^{\infty}\frac{4k}{\pi\lambda_k}\left[\frac{k^2}{4\lambda_k^2 - 1}\operatorname{Re}\psi'(\lambda_k + iZ\alpha) - \frac{\lambda_k + k^2}{2k^2(2\lambda_k + 1)}\right], \tag{7}$$

where $\psi'(z) = d\psi(z)/dz$. Expanding in $Z\alpha$ yields the corresponding portions of Eqs. (2) and (3).

The next term in the potential is given by

$$a_{2\lambda} = \frac{8\alpha(Z\alpha)}{2\lambda(2\lambda+1)}\frac{\Gamma(-2\lambda)}{\Gamma(2\lambda+1)}\frac{\cos\pi\lambda}{\pi}\int_m^{\text{“}\infty\text{”}}\frac{dq}{m}\left(\frac{2q}{m}\right)^{2\lambda}\frac{\epsilon}{q}\left|\frac{\Gamma(\lambda + iZ\alpha\epsilon/q)}{\Gamma(1 + iZ\alpha\epsilon/q)}\right|^2, \tag{8}$$

where $\epsilon^2 = q^2 - m^2$, and where here $\lambda = \lambda_1 = [1 - (Z\alpha)^2]^{1/2}$. The integral taken literally diverges; the upper limit "∞" means that the integrand is to be expanded in powers of m^2/q^2. The terms of order $(m^2/q^2)^0(2q/m)^{2\lambda}$ and $(m^2/q^2)^1(2q/m)^{2\lambda}$ diverge and are to be integrated formally, dropping the contribution of the upper limit. The remaining terms converge and are to be integrated over the full range $m < q < \infty$. This prescription arises from a careful treatment of the appropriate Mellin-Barnes representations of the Green's functions. The expansion of Eq. (8) in powers of $Z\alpha$ yields the corresponding portions of Eqs. (2) and (3), with the $\ln mr$ term arising from expanding $(mr)^{2\lambda}$.

At $\lambda = \frac{1}{2}$, both Eqs. (7) and (8) have singularities. Moreover, both terms are then of order mr. It is straightforward to show that the singularities cancel. The pole of a_1 arises from the first term in the k sum:

$$a_1^{\text{pole}} = \alpha(Z\alpha)2\pi^{-1}\operatorname{Re}\psi'(\tfrac{1}{2} + i\sqrt{3}/2)(\lambda - \tfrac{1}{2})^{-1} = \alpha(Z\alpha)[\pi/(\cosh\tfrac{1}{2}\pi\sqrt{3})^2](\lambda - \tfrac{1}{2})^{-1}. \tag{9}$$

It is convenient to separate out this pole:

$$a_1 = \alpha Z\alpha/\pi + \alpha(Z\alpha)^3\{a_1^{\text{reg}} + \tfrac{4}{3}[\pi/(\cosh\tfrac{1}{2}\pi\sqrt{3})^2](\lambda - \tfrac{1}{2})^{-1}\}, \tag{10}$$

$$a_{2\lambda} = -\tfrac{2}{9}\alpha Z\alpha + \alpha(Z\alpha)^3\{a_{2\lambda}^{\text{reg}} - \tfrac{4}{3}[\pi/(\cosh\tfrac{1}{2}\pi\sqrt{3})^2(\lambda - \tfrac{1}{2})^{-1}\}. \tag{11}$$

The functions a_1^{reg} and $a_{2\lambda}^{\text{reg}}$ are shown in Figs. 2(a) and 2(b). They are slowly varying except for a singularity as $Z\alpha \to 1$, a singularity which cancels in the complete potential.

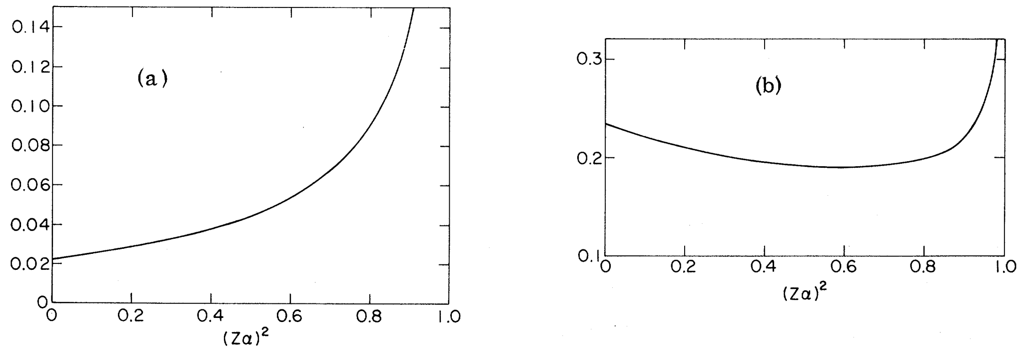

FIG. 2. (a) The regular portion of the coefficient in the potential of order $m(mr)$, a_1^{reg}, as a function of $(Z\alpha)^2$ [cf. Eq. (10)]. (b) The regular portion of the coefficient in the potential or order $m(mr)^{2\lambda}$, $a_{2\lambda}^{\text{reg}}$, as a function of $(Z\alpha)^2$ [cf. Eq. (11)].

A physical system of considerable interest is muonic ^{208}Pb. The muonic x ray associated with the transition $5g \to 4f$ has been the subject of considerable scrutiny[5] because of the discrepancy between theory and experiment, $E_{\text{th}} - E_{\text{exp}} \cong 53 \pm 20$ eV, typical of a number of muonic x-ray transitions in heavy atoms. The effect of our new results is best seen by subtracting the Uehling piece to yield a potential of orders $(Z\alpha)^3$ and greater:

$$V_1^{(3+)}(r) = m(mr)[a_1 - \alpha Z\alpha/\pi], \tag{12}$$

$$V_1^{(3+)}(r) = m(mr)^{2\lambda}a_{2\lambda} - [-\tfrac{2}{9}\alpha Z\alpha m(mr)^2]. \tag{13}$$

The previously available result was the $(Z\alpha)^3$ approximation given in Eq. (3):

$$V_1^{(3)}(r) = \alpha(Z\alpha)^3\pi^{-1}m(mr)[-6\zeta(3) + \tfrac{1}{16}\pi^4 + \tfrac{1}{6}\pi^2], \tag{14}$$

$$V_2^{(3)}(r) = \alpha(Z\alpha)^3 m(mr)^2[\tfrac{2}{9}\ln 4mr + \tfrac{2}{9}\gamma + \tfrac{2}{3}\zeta(3) - \tfrac{31}{27}]. \tag{15}$$

A comparison of these expressions is given in Table I. Although the modifications in going from Eq. (14) to Eq. (12) and from Eq. (15) to Eq. (13) are large, the net effect on the x-ray transition energy is small. Thus the discrepancy between theory and experiment for this and similar transitions is substantially unchanged.

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[1]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).

[2]J. Blomqvist, Nucl. Phys. B48, 95 (1972).

[3]T. L. Bell, Phys. Rev. A 7, 1480 (1973).

[4]L. S. Brown, R. N. Cahn, and L. D. McLerran, to be published.

[5]J. Arafune, Phys. Rev. Lett. 32, 560 (1974); L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. Lett. 32, 562 (1974); M. Gyulassy, Phys. Rev. Lett. 32, 1393 (1974).