^{3}He A and the rather sudden appearance of good quality ringing in ^{3}He B only very near to T_c reflect the known flow properties of the two phases.[14] The chemical potential differences in space produced by ΔH in our rather inhomogeneous geometry are comparable to those due to ΔT in heat-flow experiments which drive ^{3}He A supercritical over a wide temperature range in ^{3}He A, and ^{3}He B supercritical only very near T_c. The tendency to "stir" the system by the flow of magnetization supercurrents[15,5] is thus inhibited in ^{3}He A but not inhibited in ^{3}He B except very near T_c. Experiments to test this possibility are now being considered.

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Coupled Nonlinear Electron-Plasma and Ion-Acoustic Waves

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We obtain solutions describing stationary one-dimensional propagation of a coupled nonlinear electron-plasma wave and a nonlinear ion-acoustic wave. These waves have amplitudes linearly proportional to one another, and propagate with approximately the ion-acoustic velocity in the form of periodic wave trains, including solitary waves as a special case.

Nonlinear stationary propagation of plasma waves has been investigated extensively in recent years.[1-4] One-dimensional propagation of small- but finite-amplitude ion-acoustic waves in a collisionless cold-ion plasma is described by a Korteweg–deVries equation,[5] and the theoretical prediction of steepening and soliton formation has been confirmed by experiments.[6] A long-wavelength electron-plasma wave obeys a nonlinear Schrödinger equation.[7] Its stationary solutions in the one-dimensional case include envelope soliton, periodic wave train, and finite-amplitude plane wave. The latter is subject to a modulational instability under certain conditions.

In this paper, we present some special solutions which describe coupled, stationary, one-dimensional propagation of a nonlinear electron wave and a nonlinear ion wave. The basic equations are the Schrödinger equation for the electron wave, with a potential proportional to the ion-density perturbation, and the cold-ion fluid equations for the ion wave, supplemented by the electron-pressure-balance equation. Our solutions have the form of periodic wave trains, including solitary waves as a special case, and have the following properties: (i) Both electron and ion waves move with a group velocity very close to the ion-acoustic velocity C_s, and (ii) the amplitudes of the two waves are proportional to each other.

Our solution appears to be of particular importance in the nonlinear stage of parametric instabilities due to an electron-plasma wave acting as the pump. The basic physical mechanism involved is the same as for the oscillating two-stream and modulational instabilities.[7] Namely, the ponderomotive force associated with the electron wave induces an ion-density perturbation which in turn traps the electron wave. The difference lies in the consideration of the dynamical and resonantly enhanced ion response to the ponderomotive force. In the usual linear stability theories, the ion response is assumed to be static and hence is very small, being of second order in the pump amplitude. In our case, the ion perturbation moves with the ion-acoustic speed while trapping the electron wave, and thereby the ion response to the ponderomotive force is resonantly enhanced, becoming of first order in the pump amplitude and hence nonlinear. The importance of such a resonance effect in the modulational instability has been pointed out by Hasegawa[8] and our solution can be regarded as describing a final nonlinear stage of his stimulated modulational instability.

The following equation gives an adequate description of one-dimensional propagation of a small- but finite-amplitude, long-wavelength electron-plasma wave[10]:

$$\frac{\partial^2 u_e}{\partial t^2} - 3v_e^2\frac{\partial^2 u_e}{\partial x^2} + \omega_{pe}^2\left(1+\frac{\delta n_e}{n_0}\right)u_e = 0, \tag{1}$$

where u_e, v_e, ω_{pe}, n_0, and δn_e are, respectively, the fluid velocity, thermal velocity, plasma frequency, average density, and low-frequency density perturbation of the electron. We write

$$u_e(x,t) = \tilde{u}_e(x,t)e^{-i\omega_0 t} + \tilde{u}_e^*(x,t)e^{i\omega_0 t},$$

and assume that $\omega_0 \simeq \omega_{pe}$ and $\tilde{u}_e$ is slowly varying in time. We then neglect $\partial^2\tilde{u}_e/\partial t^2$ and approximate $\omega_0^2 - \omega_{pe}^2$ by $2\omega_{pe}\Delta$, where $\Delta = \omega_0 - \omega_{pe}$. From now on, we use ω_{pe}^{-1} and $\lambda_D = v_e/\omega_{pe}$ as units of time and length and denote the dimensionless variables $\tilde{u}_e/v_e$, Δ/ω_{pe}, and $\delta n_e/n_0$ simply by $\tilde{u}_e$, Δ, and δn_e. Equation (1) then becomes

$$i\,\partial\tilde{u}_e/\partial t + \tfrac{3}{2}\,\partial^2\tilde{u}_e/\partial x^2 + (\Delta - \tfrac{1}{2}\,\delta n_e)\tilde{u}_e = 0. \tag{2}$$

As will be shown later, δn_e depends only on the amplitude of $\tilde{u}_e$ and not on its phase. Then, if $\tilde{u}_e = w(x,t)$ is a solution of (2), any function produced by the following transformation is also a solution:

$$w(x - x_0 - Vt, t)\exp(\tfrac{1}{3}iVx - \tfrac{1}{6}iV^2t + i\theta), \tag{3}$$

where V, x_0, and θ are arbitrary parameters. Keeping this in mind, we look for a stationary solution, $w(\xi)$, which satisfies

$$\tfrac{3}{2}\partial^2 w(\xi)/\partial\xi^2 + [\Delta - \tfrac{1}{2}v(\xi)]w(\xi) = 0, \tag{4}$$

where $\xi = x - x_0 - Vt$ and we put $\delta n_e = v(\xi)$ which is also assumed to be stationary.

For a low-frequency perturbation, we can neglect the electron inertia, obtaining from the electron equation of motion

$$\partial|\tilde{u}_e|^2/\partial x = (\partial/\partial x)[\varphi - \ln(1+\delta n_e)], \tag{5}$$

where φ is the low-frequency potential measured in units of T/e, T being the electron temperature and $-e$ the electron charge. The left-hand side describes the ponderomotive force. We combine this equation with the ion equations of continuity and motion,

$$\epsilon^{-1}\partial\,\delta n_i/\partial t + (\partial/\partial x)[(1+\delta n_i)u_i] = 0, \tag{6}$$

$$\epsilon^{-1}\partial u_i/\partial t + u_i\,\partial u_i/\partial x + \partial\varphi/\partial x = 0, \tag{7}$$

and the Poisson equation,

$$\partial^2\varphi/\partial x^2 = (\delta n_e - \delta n_i), \tag{8}$$

where ϵ^2 is the electron-to-ion mass ratio, δn_i the ion-density perturbation normalized by n_0, and u_i the ion fluid velocity normalized by $C_s = \epsilon v_e$. The ion temperature is neglected in (7). Since we are interested in the stationary solution moving with velocity V, we can replace $\partial/\partial t$ by $-V\partial/\partial\xi$ and $\partial/\partial x$ by $\partial/\partial\xi$.

If we make the linear approximation, we obtain from (6) and (7),

$$\delta n_i = \epsilon u_i/V = \epsilon^2\varphi/V^2. \tag{9}$$

If in addition we assume local charge neutrality, $\delta n_e = \delta n_i$, we get from (5) and (9)

$$\delta n_e = |\tilde{u}_e|^2(V^2/\epsilon^2 - 1)^{-1}. \tag{10}$$

Substitution of (10) into (2) yields the usual nonlinear Schrödinger equation for the long-wavelength electron wave. It is modulationally unstable when the group velocity is subsonic (i.e., The linear approximation breaks down if V is very close to ϵ. In order to derive an appropriate nonlinear equation, we differentiate (8) with respect to ξ and add the result to the sum of (7) and ϵ/V times (6). Using (5) and keeping the terms up to second order in δn_e, one obtains

$$(\partial/\partial\xi)\{(\epsilon/V)[(1-V^2/\epsilon^2)u_i+\delta n_i u_i]+\tfrac{1}{2}u_i^2+|\tilde{u}_e|^2-\tfrac{1}{2}(\delta n_e)^2+\partial^2\varphi/\partial\xi^2\}=0. \quad (11)$$

This equation contains only small terms, either nonlinear or linear with higher derivative or small coefficient $1-V^2/\epsilon^2$. One can therefore use the linear relations (9) as well as the local charge neutrality, $\delta n_e=\delta n_i=v(\xi)$. Also, V/ϵ may be replaced by unity except for the term $V-\epsilon$. Equation (11) can then be reduced to the form

$$\frac{\partial^2 v}{\partial\xi^2}-2\lambda v+v^2+|w|^2+W=0, \quad (12)$$

where we replaced $|\tilde{u}_e|^2$ by $|w|^2$, λ is the excess Mach number, $\lambda=(V-\epsilon)/\epsilon$, and W is the integration constant. Equations (4) and (12) are our basic equations.

We expect a solution for which $|w|$ and $|v|$ are of comparable order. It is then natural to assume the form

$$|w|^2=a+bv+cv^2. \quad (13)$$

Equations (12) with (13) have a general solution expressible in terms of Jacobi's elliptic function $\mathrm{cn}(\alpha\xi;k)$[11]:

$$v=v_0+A\,\mathrm{cn}^2(\alpha\xi;k), \quad (14)$$

where v_0 is determined by the condition that the spatial average of v should vanish,

$$v_0=-[A/2\alpha K(k)]\int_0^{2K}dx\,\mathrm{cn}^2(x;k), \quad (15)$$

with $K(k)$ being the complete elliptic integral of the first kind. The other constants are to be determined such that the coefficient of each power of $\mathrm{cn}^2(\alpha\xi;k)$ vanishes in (12). There are three such relations.

Our next procedure is to separate w into amplitude and phase by writing $w=R^{1/2}e^{i\Psi}$; since v, as a solution of (13), is a function of R only, one can easily find two integrals of (4) as

$$R\,d\Psi/d\xi=M=\text{const}, \quad (16)$$

$$R^{-1}[(dR/d\xi)^2+\tfrac{8}{3}(\Delta+b/4c)R^2-(R/9c^2)(b+2cv)^3+4M^2]=E=\text{const}. \quad (17)$$

Substituting (14) into (13) and then into (17) gives an algebraic equation for $\mathrm{cn}^2(\alpha\xi;k)$ in fifth power. Setting the coefficient of each power equal to zero, we get six relations, of which only five are found to be independent. For a given value of Δ, there are twelve independent parameters: a, b, c, A, α, k, λ, W, M, E, x_0, and θ or $\Psi(\xi=0)$. Of these, we can determine only eight parameters, four being left free to be chosen.

Particularly simple solutions are obtained in the case $M=0$, i.e., $d\Psi/d\xi=0$. In this case, we find a solution in the form

$$w=B\,\mathrm{cn}(\alpha\xi;k)\,\mathrm{sn}(\alpha\xi;k) \quad (18)$$

with

$$A=-18k^2\alpha^2,\quad B=(432)^{1/2}k^2\alpha^2,\quad \alpha^2=\tfrac{2}{3}(5k^2-4)^{-1}(\tfrac{1}{2}v_0-\Delta),$$

$$k^2-1=[(v_0-2\lambda)v_0+W]/2A\alpha^2,\quad -\lambda+v_0=2\alpha^2(1+4k^2).$$

Clearly, $|A|$ and $|B|$ are of the same order, so are $|w|$ and $|v|$. The general form of (14) and (18) describes a periodic wave train with three parameters, W, x_0, and θ, being left free to be chosen. In the special case in which $k^2=1$, the period of the wave train becomes infinite and the solution is reduced to a solitary wave. In this case, $K\to\infty$ and hence $v_0\to 0$, so that W must be zero. The explicit form of the solution is

$$v(\xi) = 12\Delta\,\mathrm{sech}^2[(-2\Delta/3)^{1/2}\xi], \tag{19}$$

$$w(\xi) = (192)^{1/2}\Delta\,\mathrm{sech}[(-2\Delta/3)^{1/2}\xi]\tanh[(-2\Delta/3)^{1/2}\xi]. \tag{20}$$

Since Δ has to be negative, this solution can exist only in the overdense region ($\omega_0 < \omega_{pe}$). The density perturbation $v(\xi)$ is negative, implying a density depletion, but is of the same order as $|w(\xi)|$. Since $|v(0)|$ is 12 times as large as $|\Delta|$, the plasma is locally underdense and the electron wave is trapped in that region. The excess Mach number λ is negative, i.e., subsonic, and is given by $20\Delta/3$. Whereas the ion-density perturbation is symmetric around $\xi = 0$, the electron wave is antisymmetric and shows a phase jump at $\xi = 0$. For given Δ, the only free parameters are the initial position x_0 and the initial phase θ, all the other parameters being uniquely determined by Δ.

Let us finally discuss the effect of Landau damping. First, the ion Landau damping due to a finite ion temperature prevents a sharp resonance at $V = C_s/v_e = \epsilon$, $|\lambda|$ $(= |V - \epsilon|)$ becoming at least of order ν_i/ω_s, where ν_i and ω_s are the damping rate and the frequency of the ion-acoustic wave. On the other hand, a large ion-density perturbation (of order $|v| \sim |w|$) predicted by the present theory assumes $|\lambda|$ to be of order $|w|$ or less; otherwise, $|v|$ becomes much smaller, being of order $|w|^2$. This implies that in the case when $|w|$ acts as a pump the ion Landau damping brings in a threshold ($|w| > \nu_i/\omega_s$) for the occurrence of a large ion-density perturbation. Secondly, the Landau damping of the electron wave will cut down the large-wave-number components and thereby will tend to smooth the perturbation. Finally, a large ion-density perturbation will benefit the ion heating as compared with the usual parametric instabilities where only electrons are selectively heated.[12] However, this ion heating will eventually destroy the present solution by increasing the threshold.

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