Two-Dimensional Stability of Langmuir Solitons*

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We report on the stability of plane Langmuir solitons to perturbations in the transverse direction and of cylindrically symmetric solitons to azimuthal perturbations. Two-dimensional numerical solutions of the basic equations, including the effect of finite ion inertia, show that both cases are unstable and that the collapse of annular solitons in the azimuthal direction is faster than their radial collapse.

Plasma heating by short pulses from high-power sources, e.g., intense relativistic electron beams[1] or lasers,[2] takes place in general via collective effects for a rather broad regime of parameters. The high power of these sources precludes in many cases the application of the theories of weak turbulence.[3] Numerical simulations[4,5] of electron-beam and laser interactions with plasma have clearly indicated that the energy of the high-frequency $\omega \sim \omega_e$, long-wavelength $\lambda \sim c/\omega_e$ (ω_e is the plasma frequency) primary spectrum generated by the two-stream instability is rapidly (~a few hundred ω_e^{-1}) transferred to short wavelengths $\lambda \gtrsim \lambda_{\rm D}$, where $\lambda_{\rm D}$ is the Debye length. This observation is in direct conflict with the conclusions from weak-turbulence theory[3] which demand that nonlinear processes should transfer the energy of the primary spectrum to still longer wavelengths. Explanations for these numerical results have been offered in terms of a parametric instability driven by the primary spectrum.

The spectrum observed in numerical simulation studies, $|\vec{E}_k|^2 \propto k^{-2}$, has recently been interpreted[6] to be equivalent to treating the turbulent fields as a random interaction of "solitons" composed of Langmuir fluctuations. In this approach, the "soliton", which is a condensation of high-frequency energy localized by creating a well in the plasma density of magnitude $\delta n/n_0 \approx -|\vec{E}_k|^2/4\pi T_e$, is regarded as a basic element. It thus becomes necessary to investigate the stability of solitons and the nature of their interaction with one another.

In one dimension, solitons are stable entities. Our investigation of the interaction of two solitons using particle simulation codes as well as fluid codes[7] confirms the conclusions of Degtyarev, Makhamkov, and Rudakov[8]: viz, (i) if ion inertia is neglected then two solitons pass through each other and emerge without a change of shape or amplitude; (ii) inclusion of ion inertia allows the possibility of energy being either radiated or absorbed as ion sound waves. Thus two solitons can coalesce under certain conditions to form a single soliton with the excess energy being radiated away.

In this Letter we report on the stability of solitons to perturbations in two dimensions. The fundamental equations for the nonlinear interaction of high-frequency electron oscillations with the ion fluid are due to Zakharov[9]:

$$\nabla\cdot(i\,\partial\vec{E}/\partial t + \nabla\nabla\cdot\vec{E} - \nu\vec{E}) = 0, \tag{1}$$

$$\partial^2\nu/\partial t^2 - \nabla^2\nu = \nabla^2|\vec{E}|^2, \tag{2}$$

where $\vec{E}$ is the complex amplitude of the high-frequency electric field, $\vec{\mathcal{E}} = \vec{E}(\vec{x}, t)\exp(-i\omega_e t)$, and ν is the low-frequency perturbation in the ion density. Equations (1) and (2) are in dimensionless units; the units of time, space, electric field, and density are, respectively, $\frac{3}{2}(m_i/m_e)\times\omega_e^{-1}$, $\frac{3}{2}(m_i/m_e)^{1/2}\lambda_{\rm D}$, $(64\pi/3)^{1/2}(m_e/m_i)^{1/2}(n_0 T_e)^{1/2}$, and $\frac{4}{3}(m_e/m_i)n_0$. These equations admit several invariants. The first two invariants are $I_1 = \int|\vec{E}|^2 \times d^3r$, and $I_2 = \int[\nu|\vec{E}|^2 + |\nabla\cdot\vec{E}|^2 + (\nu^2 + |\vec{v}|^2)/2]\,d^3r$, where $\partial\nu/\partial t + \nabla\cdot\vec{v} = 0$ and the second invariant may be written as $I_2 = \int(|\nabla\cdot\vec{E}|^2 - |\vec{E}|^4/2)\,d^3r$ when the ion-inertia term of Eq. (2) is neglected.[9]

We treat two cases: (i) the stability of a plane one-dimensional soliton,[10] $\vec{E}(x, t) = \hat{x}E_0\exp[i(kx - \omega t)]/\cosh[k_0(x - v_g t)]$, $\nu(x, t) = |\vec{E}(x, t)|^2/(v_g^2 - 1)$, with $E_0 = \sqrt{2}k_0(1 - v_g^2)^{1/2}$, $\omega = k^2 - k_0^2$, $v_g = 2k < 1$, to perturbations in two dimensions, and (ii) the stability of a cylindrically symmetric annular soliton to azimuthal perturbations.

The fundamental equations are solved numerically in two dimensions by Fourier transforms with respect to x and y, assuming periodic boundary conditions with periodicity lengths L_x and L_y. With $\psi = -\nabla\cdot\vec{E}$ and the use of Fourier transforms, Eqs. (1) and (2) yield $[\dot{\psi}]_{\vec{k}} = -ik^2[\psi]_{\vec{k}} - \vec{k}\cdot[\nu\vec{E}]_{\vec{k}}$ and $[\ddot{\nu}]_{\vec{k}} = -k^2[\nu]_{\vec{k}} - k^2[|\vec{E}|^2]_{\vec{k}}$. The dots denote time differentiation and the square brackets denote Fourier transformation with $\vec{k} = (2\pi m/L_x)\hat{x} + (2\pi n/L_y)\hat{y}$. Modes corresponding to $-m_{\max} \leq m \leq m_{\max}$ and $-n_{\max} \leq n \leq n_{\max}$ are retained in the computations. The convolution sums required in the nonlinear terms are computed by transforming back to x-y space after the addition of zero modes to eliminate the periodicity in $\vec{k}$ space. The spatially homogeneous field $[\vec{E}]_{k=0}$ which cannot be computed from $[\psi]_{\vec{k}}$ is found from the equation $[\dot{\vec{E}}]_0 = -i[\nu\vec{E}]_0$ which follows from Eq. (1). An implicit time step is used in which the linear terms are integrated exactly in time and the nonlinear terms are evaluated to order Δt^2. All spatial differentiations are carried out in $\vec{k}$ space and representation in x-y space is used only in evaluating convolutions or for diagnostic purposes. This method is similar but not identical to the split-time-step Fourier method.[11] The accuracy of the solutions has been verified by checking the variations of I_1 and repeating a number of computations with different values of $m_{\max}$, $n_{\max}$, and Δt.

The first study considers one-dimensional solitons, with $\vec{E}(x, t)$ in the x direction as defined above, which are given an initial perturbation in the y direction. The evolution in time of such a soliton with $k_0 = 2$, $k = 0$ (i.e., a standing soliton), and an initial perturbation potential $\delta\varphi = \epsilon E_x(x, 0)\exp(ik_y y)$, with $\epsilon = 0.05$, $k_y = \pi/6$, and $\delta\nu = 0$, is shown in Fig. 1. From $t = 0$ to $t = 3$, the soliton in this case remains stable while emitting ion sound waves as shown by the plots of ν at $t = 3$. However, for $t > 3$ the soliton becomes unstable, collapses, and at $t = 6.5$ most of its energy is concentrated in two blobs. This computation, with identical initial conditions, was repeated with the neglect of the ion-inertia term in Eq. (2) from which it follows that $\nu = -|\vec{E}|^2$. When ion inertia is neglected, the soliton is unstable from the initial time and collapses in a shorter time, $t = 3.5$. These computations were carried out with $L_x = L_y = 12$, $m_{\max} = n_{\max} = 10$, and $\Delta t = 0.0125$, and were repeated with $m_{\max} = n_{\max} = 20$ without significant changes in the results.

Two additional sets of computations were carried out with planar solitons to investigate the effect of k and k_y on the instability. In these computations a perturbation $\delta k_0 = \epsilon k_0 \cos(k_y y)$ on the parameter k_0 is introduced with $k_0 = 2$ and $\epsilon = 0.05$. In the first set we consider $k_y = \pi/8$ and three values $k = 0$, 0.2, and 0.3. These cases were unstable from $t = 0$ and the solitons collapsed, their maximum energy $|\vec{E}|_{\max}^2$ increasing by a factor of 4 from $t = 0$ to $t = 1.5$, 1.7, and 2.0, respectively. These results show that solitons become less unstable as their group velocity increases. In the second set of computations we assumed $k = 0$ and three values $k_y = \pi/8$, $\pi/4$, and $\pi/2$ corresponding to modes $n = \pm 1$ with $L_y = 2\pi/k_y$. For $k_y = \pi/8$ and $\pi/4$ the solitons were unstable, $|\vec{E}|_{\max}^2$ increasing by a factor of 4 from $t = 0$ to $t = 1.5$ and 1.8, respectively. For $k_y = \pi/2$ the soliton was stable out to $t = 8$. These results show that long-wavelength perturbations are most unstable.

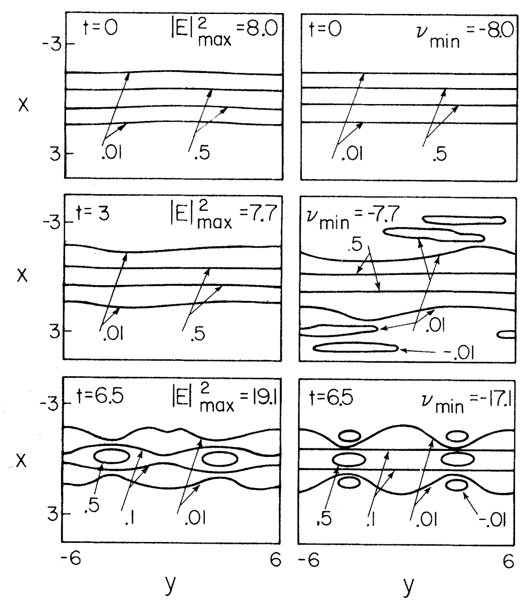

FIG. 1. Collapse of one-dimensional plane soliton perturbed in the transverse direction. The contour lines are labeled in fractions of $|\vec{E}|_{\max}^2$ and $\nu_{\min}$.

In the second study we consider the evolution of cylindrically symmetric solitons. In this case, with the neglect of ion inertia, the second moment $A = \int r^2|\vec{E}|^2 dx\, dy$ satisfies $d^2A/dt^2 < 8I_2$, whence $dA/dt < [dA/dt]_{t=0} + 8I_2 t$, which shows that the soliton collapses radially when $I_2 < 0$.[9] Computations have been carried out with a radial electric field initially of the form $E_r = E_0 \exp[ik(r - r_0)]/\cosh[k_0(r - r_0)]$, with $E_0 = \sqrt{2}k_0(1 - 4k^2)^{1/2}$, $k_0 = 2$, $r_0 = 2$, and for two cases $k = 0.125$ and $k = 0.3$. These initial conditions yield $I_2 \simeq -101$ for $k = 0.125$ and $I_2 \simeq -13$ for $k = 0.3$, and computa-

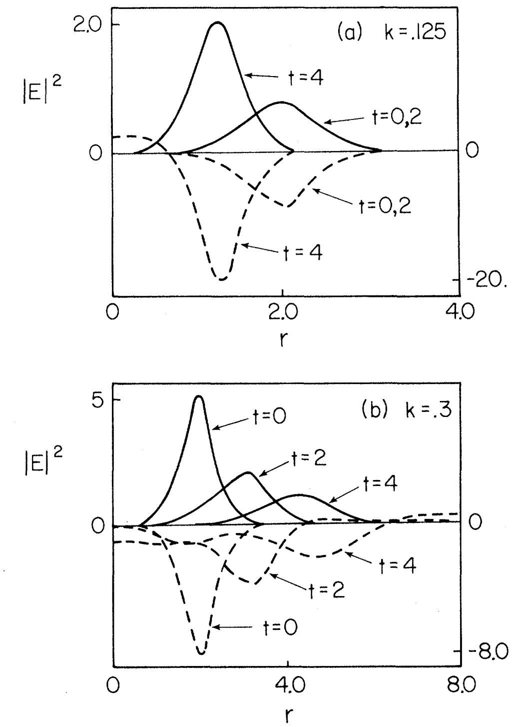

FIG. 2. Evolution of cylindrically symmetric annular solitons with $|\vec{E}|^2$ as a function of radial distance (solid lines) and as a function of ν (broken lines).

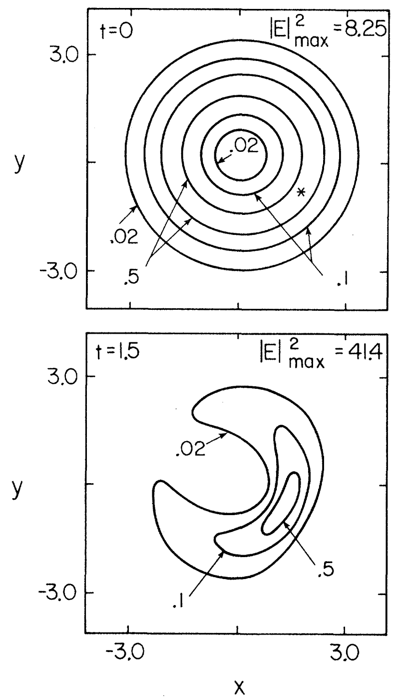

FIG. 3. Evolution of annular soliton with azimuthal perturbation with $k_0=2$, $k=0.125$, and $l=1$. The contour lines are labeled in fractions of $|\vec{E}|_{max}{}^2$. The star indicates the position of the initial maximum of $|\vec{E}|^2$.

tions with the ion-inertia term neglected in Eq. (2) confirm that the solitons collapse in both cases with collapse times $t\simeq 1.5$ for $k=0.125$ and $t\simeq 4$ for $k=0.3$. The results of computations which include the ion inertia are shown in Fig. 2. For $k=0.125$ the soliton now remains approximately stationary until $t=2.5$ after which its radius decreases, while for $k=0.3$ the radius expands to twice its initial radius from $t=0$ to $t=4$.

We now examine the evolution of cylindrical solitons when an azimuthal perturbation of the form $\delta E_r = \epsilon E_r \cos(l\varphi + \varphi_0)$ is superimposed over the cylindrically symmetric field E_r. The result of a computation with $k=0.125$ corresponding to the slowly collapsing soliton of Fig. 2, with $\epsilon=0.05$, $l=1$, and $\varphi_0=\pi/3$, is shown in Fig. 3. We observe that this case is unstable and the soliton collapses in the azimuthal direction within a time $t\simeq 1.5$, i.e., much faster than the radial collapse. Further computations with $l=2$ and 3 resulted in slower azimuthal rates of collapse, with collapse times $t\simeq 2.25$ and 3.0, respectively. Finally we considered the case $k=0.3$ with $l=2$. The soliton in this case expands radially but also collapses in the azimuthal direction and after time $t=2.5$ its energy becomes concentrated in two blobs. The cylindrical computations were carried out with $(m^2+n^2)^{1/2}<20$ and values of $L_x=L_y$ ranging from 8 to 16.

We have shown that plane stable solutions of Eqs. (1) and (2) are unstable to perturbations in the transverse direction. Similarly, cylindrically symmetric annular solutions collapse in the azimuthal direction when the constraint of cylindrical symmetry is relaxed. In all cases considered the collapse in the azimuthal direction occurs faster than the radial collapse. The high-frequency electric field energy appears to be channeled into regions where the field energy is already high and to condense into ever smaller and denser blobs. This behavior suggests that in the long-time limit the energy distributions must depend on particle interactions or other physical effects[12] not included in Eqs. (1) and (2).

During the course of these investigations we were made aware of the work of Degtyarev, Zakharov, and Rudakov.[13] With respect to plane solitons their conclusions are similar to those presented in this Letter. We have extended these results to the stability of cylindrically symmetric solitons to azimuthal perturbations.

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Operation of a Polarized ^{3}He Source

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A source of polarized $^3He^{++}$ particles based on the Lamb shift in $^3He^+$ ions is now operating. After axial injection into a cyclotron and acceleration to 33 MeV the on-target beam intensity in a scattering chamber exceeds 0.1 nA. Nuclear scattering experiments together with a study of the source characteristics indicate a beam polarization value of $P=0.40\pm0.05$.

The feasibility of producing a beam of $^3He^{++}$ ions with nuclear polarization was briefly reported in an earlier communication[1] and a proposed design for a polarized source has also been outlined.[2] This source, which is the first of its kind, has now been built and is currently used in nuclear reaction experiments. The present Letter describes its performance in operation with a cyclotron.

In the source, metastable $^3He^+(2S)$ ions are created by the collision of a primary $^3He^{++}$ beam with air molecules in a gas canal. The ions then pass through a strong axial magnetic field of 0.20 T in which Zeeman splitting of the $2S_{1/2}$ and $2P_{1/2}$ states takes place. The lower $2S_{1/2}$ states ($m_s = -\frac{1}{2}$, $m_I = \pm\frac{1}{2}$) are quenched to the short-lived $2P$ states by passage through a rf cavity in which a transverse electric field is excited at a frequency of 10 GHz. After entering a weak-field region in which hyperfine coupling is re-established the remaining metastable beam carries a nuclear polarization $P=0.50$. The metastable component ($2S$) of the total beam is next ionized, with high selectivity with respect to the large $^3He^+(1S)$ ground-state component, in a second gas canal containing air. There is a fuller description of these processes in Ref. 2.

Among the factors determining the nuclear polarization of the final beam is the efficiency of the electron-transfer processes in the two gas canals. Measurements[1] have shown that the overall efficiency is approximately 0.1%, with a selectivity in excess of 5:1 for $^3He^{++}$ created from $^3He^+(2S)$ compared with $^3He^{++}$ created from the ground state ($1S$). This has been confirmed by detailed measurements[3,4] of the individual cross sections for the main processes in the two gas canals.