A sketch of the proposed experiment is shown there in Fig. 9.

[4]R. C. Jaklevic, J. Lambe, J. E. Mercereau, and A. H. Silver, Phys. Rev. 140, A1628 (1965). The analogous experiment is the one in which they produced phase modulation by motion of superconducting electrons. On a base film they located two weak-link junctions which were connected electrically in parallel but separated 8 mm apart. By passing a drift current through the base, they observed a periodic maximum Josephson current through the junctions. Our channel velocity $v_{ch}$ and filling rate in $B$ are analogous to their drift current and maximum Josephson current, respectively.

[5]An alternative to the quantum mechanical explanation is to assume *a priori* that circulation is quantized and then pursue classical fluid dynamics.

[6]Using similar phase-coupling arguments, H. E. Corke and A. F. Hildebrandt, Phys. Rev. A 2, 1492 (1970), explained observed restrictions in the flow of helium.

[7]The phase difference is largest in the neighborhood of the narrowest point in the orifice. From Fig. 4(a) we see that the opening is actually located at the bottom of a deep blind hole. Thus the wall thickness around the narrow opening is only a few microns thick.

[8]To date we have been unsuccessful in detecting the vortices individually but further tests are planned.

[9]See, for example, Ref. 3 or W. Zimmerman, Jr., Phys. Rev. Lett. 14, 976 (1965). Experimentally this was observed by R. Carey, B. S. Chandresekhar, and A. J. Dahm, Phys. Rev. Lett. 31, 873 (1973).

[10]The phase drop, $\Delta\Phi$, in the orifice needs to be $\lesssim \pi$, but since $\Delta\Phi = \int_0^L v(x)\,dx$, where $L$ is the thickness, we need to minimize the thickness of the narrowest area which is done in the orifice shown in Fig. 4(a) but not in Fig. 4(b).

[11]The effect was not observed above $T_\lambda$.

# Hydrodynamics of $^3$He in Anisotropic $A$ Phase

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The hydrodynamic theory of mass transport in $A$-$^3$He is derived from that phase's broken symmetries and thermodynamics. First, second, and fourth sound as well as orbit waves are obtained as the normal modes.

From the very active experimental[1] and theoretical[2,3] work on the low-temperature phases of $^3$He it has become clear that several symmetries are simultaneously broken in the so-called $A$ phase. One of these, broken gauge invariance, is common to all known superfluids. In addition, orbital rotational invariance is also broken in $A$-$^3$He, because of an alignment of the orbital angular momentum of the triplet pairs.[4] Since this implies a directional long-range order in real space (rather than spin space like, e.g., in ferromagnets), one is strongly reminded of the order in a nematic liquid crystal.[5] However, unlike in nematics, the directional order parameter in $A$-$^3$He (being a certain angular momentum) is ***odd*** under time reversal and transforms like an *axial* vector under spatial inversion. In this latter respect $A$-$^3$He is more similar to ferromagnets.[6]

These facts have far-reaching consequences for the low-frequency collective excitations in the $A$ phase. Clearly, low-frequency "orbit waves" must exist, which are the Goldstone excitations of the broken rotational symmetry.[7] Even more exciting, the new order parameter, being a vector in real space and odd under time reversal, can couple to the mass current, as pointed out by de Gennes.[8]

A rich literature on this subject has already grown up.[9] The results obtained have been restricted in their domain of validity, however, since they were based on gradient expansions of the Ginzburg-Landau type and/or on BCS weak-coupling theory, eventually corrected for Fermi-liquid and spin-fluctuation effects. By contrast, the present paper is concerned with the general hydrodynamics of $A$-$^3$He, which has not been given previously. This theory has the advantage of being model independent and rigorous in the hydrodynamic limit. The obvious disadvantage is the impossibility to compute the phenomenological parameters entering the hydrodynamical equations within the same approach. In the case of superfluid $^4$He the corresponding theory is the well-known two-fluid hydrodynamics[10] which has proven to be of great value. For simplicity we will disregard spin motion and consider linearized hydrodynamics only.

We begin with a statement of the five conservation laws for the mass density $\rho$, momentum density $\vec{g}$, energy density $\epsilon$, and their appropriate currents,

$$\dot{\rho}+\nabla\cdot\vec{g}=0;\quad \dot{g}_i+\nabla_j\sigma_{ij}=0;\quad \dot{\epsilon}+\nabla\cdot\vec{j}_\epsilon=0, \qquad (1)$$

which already completely describe a simple fluid in the hydrodynamic limit. In $A$-$^3$He there exist, in addition, two quasiconserved quantities, due to the broken symmetries: a "phase" $\varphi$ which does not commute with the particle number $N$, and which we normalize in such a way that $[\varphi,\frac{1}{2}N]=-i$; and an axial vector $\vec{l}$ (in the direction of the net orbital momentum of the triplet pairs) which does not commute with the angular momentum $\vec{L}$. $\varphi$ and $\vec{l}$ are assumed to be odd under time reversal. $\hat{l}$ is the unit vector in the direction of $\vec{l}$, which will be generally taken as the 3 direction. The commutator is understood as a Poisson bracket. Since we are concerned with a macroscopic theory of $A$-$^3$He, the microscopic definitions of $\varphi$ and $\vec{l}$ need not concern us.[11]

Because of the existence of $\varphi$ and $\vec{l}$, Eqs. (1) are no longer complete and have to be amended by three further equations:

$$(\hbar/2m)\dot{l}_i+X_i=0;\quad (\hbar/2m)\dot{\varphi}+\mathcal{J}_\varphi=0. \qquad (2)$$

Like in antiferromagnets or nematics, the absolute value of $\vec{l}$ is not considered a hydrodynamic variable, i.e., we assume $\hat{l}_i\dot{l}_i=0$ and $\hat{l}_i\nabla_k l_i=0$ on the hydrodynamic time and length scale. The currents in Eqs. (1) and (2) now have to be determined.

From $\varphi$ and $\vec{l}$ one can construct two different velocity fields,[8,9]

$$(2m/\hbar)\vec{V}^{(s)}=\nabla\varphi;\quad (2m/\hbar)\vec{V}^{(r)}=\nabla\times\vec{l}, \qquad (3)$$

which are curl free and source free, respectively. Consequently, we may write out the momentum density in terms of the three different components

$$\vec{g}=\overleftrightarrow{\rho}^{(n)}\cdot\vec{V}^{(n)}+\overleftrightarrow{\rho}^{(s)}\cdot\vec{V}^{(s)}+\overleftrightarrow{C}\cdot\vec{V}^{(r)}, \qquad (4)$$

where $\vec{V}^{(n)}$ is the normal-fluid velocity. The three mass-density tensors are all of the form

$$\rho_{ik}^{(n)}=\rho_\perp^{(n)}(\delta_{ik}-\hat{l}_i\hat{l}_k)+\rho_\parallel^{(n)}\hat{l}_i\hat{l}_k,\ \text{etc.} \qquad (5)$$

Since a uniform Galilean transformation leaves the field $\vec{l}$ unchanged, we must have $\rho\overleftrightarrow{1}=\overleftrightarrow{\rho}^{(n)}+\overleftrightarrow{\rho}^{(s)}$.

The remaining currents in Eqs. (1) and (2) are now determined from the broken symmetries and thermodynamics.[5,6,10] First we write down the Gibbs relation appropriate to our set of hydrodynamical variables:

$$Td(\rho s)=d\epsilon-\mu d\rho-\vec{V}^{(n)}\cdot d\vec{g}-\frac{\hbar}{2m}\vec{\lambda}^{(s)}\cdot d(\nabla\varphi)-\frac{\hbar}{2m}\phi_{ij}\,d(\nabla_j l_i). \qquad (6)$$

As usual, only the gradients of the ordered variables enter.[6] $\mu$ is the chemical potential; $\vec{\lambda}^{(s)}$ and $\phi_{ij}$ are the thermodynamically conjugate quantities of $\nabla\varphi$ and $\nabla_j l_i$, respectively, which will be determined below. Next, we insert Eqs. (1)–(4) into Eq. (6) and integrate over space. On the reversible level the resulting entropy production has to vanish, which severely limits the possible form of the currents. A further restriction comes from the fact that $\varphi$ is canonically conjugate to the particle number and thus satisfies

$$\dot{\varphi}=-\frac{2m}{\hbar}\frac{\partial\epsilon}{\partial\rho}=-\frac{2m}{\hbar}\mu,$$

which already fixes the reversible part of $\mathcal{J}_\varphi$. Galilean and time-reversal invariance and the axial symmetry of the $A$ phase then serve to pin down the remaining currents in the form

$$\vec{g}=\rho\vec{V}^{(n)}+\vec{\lambda}^{(s)};\quad \vec{j}_\epsilon{}^R=T\rho s\vec{V}^{(n)}+\mu\vec{g};\quad \mathcal{J}_\varphi{}^R=\mu;$$
$$X_i{}^R=\beta\epsilon_{ipk}\hat{l}_k\nabla_j\phi_{pj}-[\alpha_1(\delta_{ij}-\hat{l}_i\hat{l}_j)\hat{l}_k+\alpha_2(\delta_{ik}-\hat{l}_i\hat{l}_k)\hat{l}_j]\nabla_jV_k{}^{(n)}; \qquad (7)$$
$$\sigma_{ik}{}^R=p\,\delta_{ik}+\tfrac{1}{2}\nabla_j\{\hat{l}_k[\alpha_1(\phi_{ij}+\phi_{ji})+\alpha_2(\phi_{ij}-\phi_{ji})]+\hat{l}_j(\alpha_2-\alpha_1)\phi_{ik}+(i,k\to k,i)\};$$

$\beta$, $\alpha_1$, $\alpha_2$ are phenomenological parameters. $\mathcal{J}_\varphi{}^R$, $\vec{g}$, and $\vec{j}_\epsilon{}^R$ are like in any superfluid. The expression for $X_i{}^R$ is new and characteristic of $A$-$^3$He. The $\alpha_1,\alpha_2$ terms of $X_i{}^R$ would occur also in a nematic, although $X_i{}^R$ would be odd there and $\hat{l}_i$ would be even under time reversal; that property simply cancels from both sides of the equation. The $\beta$ term in $X_i{}^R$ does not occur in nematics, but has its counterpart in ferromagnets, where it is the stiffness term, which gives rise to spin waves. Here, this term will give rise to orbit waves, even when $\vec{V}^{(n)}$ vanishes, as will be seen below. The stress tensor $\sigma_{ik}{}^R$ has been chosen in a symmetrical way in order to guarantee angular momentum conservation.[5]

Let us now turn to the parameters $\vec{\lambda}^{(s)}$ and $\phi_{ij}$ introduced in Eq. (6). Our result for $\vec{g}$ requires $\vec{\lambda}^{(s)}$ to have the form $\vec{\lambda}^{(s)} = \vec{\vec{\rho}}^{(s)} \cdot (\vec{V}^{(s)} - \vec{V}^{(n)}) + \vec{\vec{C}} \cdot \vec{V}^{(r)}$.

In order to determine the form of $\phi_{ij}$ let us first consider a frame in which $\vec{V}^{(s)}$ and $\vec{V}^{(n)}$ both vanish. There we have

$$\phi_{ij}{}^{(0)} = (\hbar/2m) K_{ijkl} \nabla_l l_k , \tag{8}$$

where $K_{ijkl}$ can contain three independent constants,

$$K_{ijkl} = K_1(\delta_{jk} - \hat{l}_j \hat{l}_k)(\delta_{il} - \hat{l}_i \hat{l}_l) + K_2 \epsilon_{ijp} \hat{l}_p \epsilon_{klq} \hat{l}_q + K_3(\delta_{ik} - \hat{l}_i \hat{l}_k)\hat{l}_j \hat{l}_l . \tag{9}$$

From the symmetry of thermodynamic cross derivatives we find

$$\phi_{ij} = \phi_{ij}{}^{(0)} + (\delta_{ik} - \hat{l}_i \hat{l}_k)\epsilon_{kpj} C_{pq}(V_q{}^{(s)} - V_q{}^{(n)}) . \tag{10}$$

Thermodynamic stability requires that $K_1$, $K_2 - \frac{1}{2}K_1$, $K_3$, $\rho_{\parallel}{}^{(s)}$, $\rho_{\perp}{}^{(s)}$, $\rho_{\parallel}{}^{(s)}(K_2 - \frac{1}{2}K_1) - C_{\parallel}{}^2$, and $\rho_{\perp}{}^{(s)}K_3 - C_{\perp}{}^2$ all be positive.

Let us now turn to the dissipative parts of the currents. They are obtained under the constraints of positive entropy production and correct space- and time-inversion symmetry. The result is

$$\begin{aligned} &\vec{j}_\epsilon{}^D = -\vec{\vec{\mathcal{K}}} \cdot \nabla T ; \quad \mathcal{J}_\varphi{}^D = -\zeta \nabla \cdot \vec{\lambda}^{(s)} - \xi_{ij} \nabla_j V_i{}^{(n)} ; \\ &X_i{}^D = -\eta(\delta_{ik} - \hat{l}_i \hat{l}_k)\nabla_j \phi_{kj} - \xi_{kji} \nabla_k V_j{}^{(n)} ; \\ &\sigma_{ik}{}^D = -\nu_{ikjl}(\nabla_j V_l{}^{(n)} + \nabla_l V_j{}^{(n)}) - \xi_{ik} \nabla \cdot \vec{\lambda}^{(s)} - \xi_{ikj} \nabla_l \phi_{jl}, \end{aligned} \tag{11}$$

with

$$\xi_{ikj} = \xi(\hat{l}_k \epsilon_{ijp} + \hat{l}_i \epsilon_{kjp})\hat{l}_p .$$

We obtain thus two heat conductivities $\mathcal{K}_{\parallel}, \mathcal{K}_{\perp}$, five ordinary viscosities (like in a uniaxial crystal or nematic[5]), one viscosity $\zeta$ related to curl-free superfluid flow, and one viscosity $\eta$ related to source-free superfluid flow. In addition there are two viscosities $\xi_{\parallel}, \xi_{\perp}$ relating normal and curl-free flow, and one viscosity $\xi$ relating source-free and normal flow. There are no cross terms between the two superfluid components and between the heat flux and all three fluid velocities. A number of obvious positivity conditions for these transport coefficients follow from the positivity of entropy production, which I will not write out to conserve space.

Let us finally turn to a brief discussion of the normal-mode spectrum which follows from our equations. It is convenient to take the wave vector always in the 1-3 plane. If we neglect dissipation, the equations for $\rho$, $g_1$, $g_3$, $\varphi$, and $\rho_s$ are decoupled from $l_1$, $l_2$, and $g_2$ to lowest order in $k$ and can be solved separately for first and second sound. Neglecting the small specific heat difference $C_p - C_V$, we obtain

$$\omega_1{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \cdot k^2, \quad \omega_2{}^2 = \frac{Ts^2}{C_V}\left[\frac{\rho_{\perp}{}^{(s)}}{\rho_{\perp}{}^{(n)}} k_1{}^2 + \frac{\rho_{\parallel}{}^{(s)}}{\rho_{\parallel}{}^{(n)}} k_3{}^2\right] \tag{12}$$

for the first and second sound frequencies, respectively. Thus first sound remains isotropic (because of the fact that the stress tensor $\sigma_{ik}$ is isotropic for $k \to 0$), whereas second sound is affected by and allows study of the anisotropy of the superfluid.

Orbit waves are most easily discussed for the physically interesting case of flow in packed powders, where $\vec{V}^{(n)} = 0$ replaces the momentum balance equations. We obtain for fourth sound

$$\omega_4{}^2 = \omega_0[\omega_0 + i\zeta(k_1{}^2\rho_{\perp}{}^{(s)} + k_3{}^2\rho_{\parallel}{}^{(s)})] \tag{13}$$

with

$$\omega_0{}^2 = \left(\frac{\partial \mu}{\partial \rho}\right)_{(\rho s)} (\rho_{\parallel}{}^{(s)} k_3{}^2 + \rho_{\perp}{}^{(s)} k_1{}^2),$$

and for orbit waves

$$2\omega_{0W} = \pm\{4\beta^2\delta(\vec{k})(K_3 k_3{}^2 + K_1 k_1{}^2)k^2 - \eta^2[\delta(\vec{k})k^2 - K_3 k_3{}^2 - K_1 k_1{}^2]^2\}^{1/2} + i\eta[\delta(\vec{k})k^2 + K_3 k_3{}^2 + K_1 k_1{}^2], \tag{14}$$

where

$$\delta(\vec{k}) = \left[ K_2 k_1^2 + K_3 k_3^2 - \frac{k_1^2 k_3^2 (C_\parallel - C_\perp)^2}{\rho_\parallel^{(s)} k_3^2 + \rho_\perp^{(s)} k_1^2} \right] k^{-2}$$

is a positive, direction-dependent, effective mass density.

A few points are worth making here: Fourth sound is clearly separated from the orbital excitations in the hydrodynamic limit and thus does not lose its specific value for the measurement of $\vec{\rho}^{(s)}$. Only for $\beta^2$ sufficiently large compared to $\eta$ can an oscillatory behavior of orbit waves result. Experimental observation of the latter would thus provide direct information about the presence and the importance of such a term. For sufficiently large $\beta^2$ the dispersion relation is of the ferromagnetic-spin-wave type[6]; however, unlike in ferromagnets, the damping of orbit waves is also of order $k^2$, because of the fact that the directional order parameter of $A$-$^3$He is not a constant of the motion.

In the more general case $\vec{V}^{(n)} \neq 0$, things become more complicated. In particular, the $\alpha_1$ and $\alpha_2$ terms of $X_i^R$, Eq. (7), contribute to the oscillatory behavior of the orbit waves, as can already be seen from their analogous effect in nematics. These algebraically more involved results will be presented elsewhere, together with a more detailed account of this theory.

In conclusion I briefly compare my results with those of earlier work. The complete equations of linearized hydrodynamics, including dissipation, have been presented here, I believe, for the first time. For second and fourth sound it is legitimate to neglect dissipation for $k \rightarrow 0$. These results then reproduce or are consistent with earlier results obtained in the framework of a Landau quasiparticle picture[12,13] or in a generalized Ginzburg-Landau expansion;[8,9] for orbital waves the neglect of dissipation is not possible even for $k \rightarrow 0$, unless the relevant viscosities turn out to be small. Putting the latter equal to zero one arrives at results which are consistent with results from a generalized Ginzburg-Landau theory.[9] Unfortunately, I cannot resolve the discrepancy[14] in the result for $\vec{C}$, since both results fit the general frame set by hydrodynamics. It appears that a resolution of this discrepancy can only come from microscopic considerations. Finally, recall that I have only considered the hydrodynamics of mass motion in the present note. Similar work on the spin part of the hydrodynamics is currently in progress and will be reported elsewhere.


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[9]It is not possible to quote all relevant papers here. However, the literature may be traced from V. Ambegaokar, P. G. deGennes, and D. Rainer, Phys. Rev. A 9, 2676 (1974); P. Wölfle, Phys. Lett. 47A, 224 (1974), and Phys. Rev. Lett. 31, 1437 (1973); P. W. Anderson and W. F. Brinkman, in "Theory of Anisotropic Superfluidity of $^3$He," lectures presented at the Scottish Universities Summer School in Physics, St. Andrews, Scotland, 21 July–10 August 1973 (to be published).

[10]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959), p. 510.

[11]The prefactors $2m/\hbar$ in Eq. (2) and elsewhere are deliberately chosen so as to conform with the microscopic definition of $\varphi$ and $\vec{l}$. $\varphi$ and changes in $\vec{l}$ are, microscopically, rotation angles of the orbital part of the bivector order parameter [see Ambegaokar *et al.*, Ref. 9], and they are not unique since finite rotations are noncommuting operations, in general However, since we are only concerned with linear hydrodynamics here, we need to consider infinitesimal rotations only, which do commute and thus do not give rise to ambiguities of this kind.

[12]W. M. Saslow, Phys. Rev. Lett. 31, 870 (1973).

[13]Wölfle, Ref. 9.

[14]Contrast the results of Ambegaokar *et al.* (Ref. 9) with those of Wölfle (Ref. 9) and Anderson and Brinkman (Ref. 9).