Critical Dynamics of Ferromagnets in $6-\epsilon$ Dimensions

Shang-keng Ma*
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037

and

Gene F. Mazenko†
W. W. Hansen Laboratories of Physics, Stanford University, Stanford, California 94304

(Received 16 September 1974)

The critical dynamics of a model isotropic ferromagnet is studied in $6-\epsilon$ dimensions both above and below T_c. The renormalization group and the characteristic frequency exponent z are determined to order ϵ. Implications of the renormalization group are discussed. A direct calculation of the dynamic response function to order ϵ shows many interesting features, especially for $T<T_c$.

The dynamics of an isotropic ferromagnet near its critical temperature T_c has been of considerable theoretical interest.[1-3] Several authors have pointed out that for the dimension $d>6$ the dynamics becomes trivial.[3] For $d<6$, the understanding is not satisfactory, especially in connection with the violation of the dynamic-scaling hypothesis.[3]

The special feature of an isotropic ferromagnet is that each spin vector tends to precess around the local magnetic field, i.e., the external field plus the field produced by neighboring spins. For a phenomenological description, let the three-component vector field $\vec{S}(x,t)$ be the spin density at time t. We define our model by the equation of motion

$$\partial\vec{S}/\partial t=\lambda\vec{S}\times\vec{H}-\Gamma\nabla^2\vec{H}+\vec{\zeta}, \tag{1}$$

where $\vec{H}(x,t)$ is the local field, λ and Γ are constants, and $\vec{\zeta}(x,t)$ is a random noise simulating the effect of thermal agitation on the spins. We assume the local field to be given by the derivative of $F[\vec{S}]$, the free energy at the fixed spin configuration $\vec{S}$, and write a Ginsburg-Landau form for F:

$$\begin{aligned}&\vec{H}(x,t)=-\delta F/\delta\vec{S}(x,t),\\&F[\vec{S}]=\tfrac{1}{2}\int d^dx\,[(\nabla\vec{S})^2+r_0S^2+\tfrac{1}{2}u(S^2)^2-\vec{h}\cdot\vec{S}],\end{aligned} \tag{2}$$

where r_0 and u are constants, and $\vec{h}$ is the external field. In terms of Fourier components $\vec{S}_k(t)$ and $\vec{\zeta}_k(t)$, (1) becomes

$$\frac{\partial\vec{S}_k}{\partial t}=\lambda L^{-d/2}\sum_{k'}\vec{S}_{k+k'}\times\frac{\partial F}{\partial\vec{S}_{k'}}-\Gamma k^2\frac{\partial F}{\partial\vec{S}_{-k}}+\vec{\zeta}_k, \tag{3}$$

where L^d is the volume of the magnet and the wave vectors k are restricted to be less than a cutoff Λ. We assume that the noise ζ has the statistical property

$$\langle\zeta_k(t)\zeta_{k'}(t')\rangle=2\Gamma k^2\delta_{-k,k'}\delta(t-t'). \tag{4}$$

The model is thus defined. It is closely related to that discussed by Kawasaki and others.[1,4,5] We note the following: (a) The λ term in (1) reduces to the simple form $-\lambda\vec{S}\times\nabla^2\vec{S}$ and is closely related to the usual Heisenberg equation of motion. (b) We must keep the quartic terms in F if we are to have a stable solution below T_c. (c) We have neglected the coupling of the spin to the energy fluctuation. (d) The λ term corresponds to a streaming velocity[4] in the space of $\vec{S}_k$. The form of these streaming velocities appears to play a major role in determining the critical dynamics.[6] (e) Γ plays the role of a "bare" transport coefficient which is determined by dynamics over a short distance Λ^{-1}. (f) If $\lambda=0$ this model reduces to the time-dependent Ginsburg-Landau model with spin conservation. The effect of nonzero λ is our subject matter.

All physical quantities calculated will be functions of the set of parameters

$$\mu=(\lambda,r_0,u,h,\Gamma). \tag{5}$$

A renormalization-group (RNG)[7] transformation R_b, $b\geq 1$, transforms μ to $R_b\mu=\mu'$ according to the following prescriptions: (a) Eliminate $\vec{S}_q$ for $\Lambda/b<q<\Lambda$. This is done by solving the equations of motion for $\vec{S}_q$, substituting the solution in the remaining equations, and averaging over ζ_q. (b) Replace the remaining $\vec{S}_k(t)$ by $b^{1-\eta/2}\vec{S}_{bk}(tb^{-z})$ and L by bL'. The new equations are then written in the old form with modified parameters, which are identified as entries in $\mu'=R_b\mu$. For example, with $h=u=r_0=0$ we have, to order λ^2,

$$\Gamma' = b^{z-4}[1+(\lambda/\Gamma)^2(192\pi^3)^{-1}\ln b],$$

$$\lambda' = b^{z-1-d/2}\lambda.$$

We note that only the ratio λ/Γ approaches a fixed-point value. We are free to pick either λ or Γ arbitrarily. If we set $\Gamma=1$ we are simply choosing our unit of time.

The quantities η and z are so chosen that the fixed-point equation $R_s\mu^*=\mu^*$ has a solution. Our specification of μ in (5) is in general incomplete. After the performance of prescriptions (a) and (b) above, the new equations of motion will have additional terms of different form requiring more parameters to specify a complete μ. Fortunately such complications can be avoided when all entries in μ are small. Solutions for μ^* were found consistent with the smallness assumption to order ϵ. There are nontrivial fixed points, $\Gamma=1$,

$$\lambda^* = \pm(96\pi^3\epsilon)^{1/2}, \quad r_0^* = u^* = h^* = 0, \qquad \eta = 0, \quad z = 4-\epsilon/2 = 1+d/2, \tag{6}$$

and there is also a trivial fixed point with zero in all entries for μ^* with $z=4$, $\eta=0$. The two nontrivial fixed points are mirror images. We shall consider only one of them. For μ in the neighborhood of μ^*, we obtain the linearized recursion relations

$$\delta\lambda' = \delta\lambda b^x, \tag{7a}$$

$$r_0' = r_0 b^{1/\nu}, \tag{7b}$$

$$u' = u b^y, \tag{7c}$$

$$h' = h b^{4-\epsilon/2}, \tag{7d}$$

where

$x=-\epsilon$ for nontrivial fixed points,

$x=\epsilon/2$ for the trivial fixed point, (8)

and $\nu=\frac{1}{2}$, $y=-2+\epsilon$ for both cases. The nontrivial fixed point is stable in the sense that, for $r_0=0$, $R_b\mu$ approaches μ^* as b increases. The trivial fixed point is unstable since $\delta\lambda'$ grows as b increases. The crossover exponent φ associated with this instability is

$$\varphi = x\nu = \epsilon/4. \tag{9}$$

We shall restrict the rest of the discussion to the stable fixed point. From the transformation properties of μ, we easily deduce certain properties of physical quantities. For example, the linear response function[8] $G(k,\omega,\mu)$ satisfies

$$G(k,\omega,\mu) = b^{2-\eta}G(bk, b^{+z}\omega, R_b\mu).$$

More explicitly

$$G(k,\omega,\delta\lambda+\lambda^*, r_0, u) = b^2 G(bk, b^{+z}\omega, b^x\delta\lambda+\lambda^*, b^{1/\nu}r_0, b^y u) \tag{10}$$

for all $b\geq 1$, and $k<\Lambda/b$. Set $b=\xi\equiv|r_0|^{-\nu}=|r_0|^{-1/2}$. We obtain, for μ near μ^*

$$G(k,\omega,\mu) = \xi^2 G(k\xi, \omega\xi^{4-\epsilon/2}, \xi^{-\epsilon}\delta\lambda+\lambda^*, \mathrm{sgn}\, r_0, u\xi^{-2+\epsilon}). \tag{11}$$

The usual statement of dynamic scaling[2] follows if we neglect $\xi^{-\epsilon}$ and $u\xi^{-2+\epsilon}$, which are small when ξ is very large. However, we shall see that, even though they are small, they cannot be neglected in certain cases. Equation (10) is all that the RNG can tell us about G.

To obtain information not contained in the RNG analysis, we solved the equations of motion and obtained G to order ϵ by choosing $\lambda=\lambda^*$. This choice is for mathematical purposes and is in the same spirit as Wilson's choice of u_0 in computing exponents by perturbation theory.[7,9] From (11) we expect logarithms to appear as a result of ϵ in the exponents when G is computed by expanding in powers of ϵ. By setting $\lambda=\lambda^*$, $\delta\lambda=0$, we remove the logarithms due to $\delta\lambda\xi^{-\epsilon}$ to order ϵ. This helps in identifying the other exponents.

The calculation above T_c is straightforward, whereas the calculation turned out to be complicated below T_c because of the presence of a finite magnetization $\vec{M}$. The details of the calculation will be reported in another paper. We sketch some results here, with some brief discussion for $T<T_c$ below. We have calculated the dispersion formulas as solutions of $G^{-1}(k,\omega)=0$ in the following cases: (i) $T>T_c$ ($r_0>0$), $k\to 0$; here

$$\omega = -ik^2 r_0\{1-[\lambda^{*2}/6(64)\pi^3]\ln r_0+\ldots\}$$

$$= -ik^2 r_0^{1-\epsilon/4}. \tag{12a}$$

(ii) $T=T_c$, k small; here

$$\omega = -ik^{4-\epsilon/2}. \tag{12b}$$

(iii) $T<T_c$, longitudinal mode ($\parallel\vec{M}$), k small; here

$$\omega = -i(2\tau)k^{2-\epsilon/2}. \tag{12c}$$

(iv) $T<T_c$, transverse or spin-wave mode ($\perp \vec{M}$), k small; here

$$\omega=\lambda^*(\tau/u)^{1/2}k^2-i(\text{const}\times\tau/u)^{-\epsilon/8}k^4, \tag{12d}$$

where $\tau\equiv|r_0|=\xi^{-2}\propto T_c-T$, and the unit of time is chosen such that $\Gamma=1$.

All of these results are consistent with $z=4-\epsilon/2$, (7), and (10). The role of u in assuring this consistency is evident in the spin-wave case (12d). One may not ignore u [see (7c)] and hence the usual form of dynamic scaling appears violated. Our calculation for $G_\parallel(k,\omega)$ shows that u plays no role in $G_\parallel$ and hence the usual form of dynamic scaling applies.

A few remarks are in order. (a) The renormalization group transforms the equations of motion, but does not solve them. It is like the rotation group in atomic physics. It helps but does not tell the whole story. It is not surprising that the exponent z alone may not be sufficient to describe the characteristic frequency. (b) The result $z=1+d/2$ agrees with well-known dynamic-scaling and mode-coupling[4] arguments for $d<4$ and seems to be general. The statics for $d<4$ is of course very different from that for $d>4$. In particular, $u^*\neq 0$ and the last entry of (10) would be $b^{d-4}\delta u+u^*$ for $4-d$ small. (c) The statics for $d>4$ is often termed as "mean field" or "free field." But in our calculation we had to keep many more terms than the mean-field-theory approach would keep to arrive at (12) consistently. The statics is correctly generated by the equations of motion at least to the order calculated. The verification turned out to be nontrivial.

Here we supply a few additional details of the calculation below T_c. In this case we must take into account the nonvanishing value of S_z in the absence of $\vec{h}$. We do this by redefining our fields:

$$\varphi_z=S_z-M,$$

so that $\langle\varphi_z\rangle_{h=0}=0$, and

$$\varphi_\pm=\tfrac{1}{2}\sqrt{2}\,(S_x\pm iS_y).$$

Our equation of motion then leads to the zeroth-order ($\lambda=u=0$, $uM^2+r_0=0$) response functions,

$$G_z^0(k,\omega)=(-i\omega/\Gamma k^2+k^2-2r_0)^{-1},$$

$$G_\pm^0(k,\omega)=-[i(\omega\mp M\lambda k^2)+\Gamma k^4]^{-1}(\Gamma k^2\pm i\lambda M).$$

The dispersion relations given by (12) result from choosing $\lambda=\lambda^*$ and self-consistently (renumbering $M\sim u^{-1/2}$) calculating corrections to G^0 to second order in λ^2 and to first order in u.

We thank Dr. B. I. Halperin and Dr. C. Hohenemser for helpful conversations. One of us (G.F.M.) is happy to thank Professor S. Doniach for support and encouragement. He also thanks the National Science Foundation for support during his visit to the University of California at San Diego, where much of this work was accomplished.

---

*Alfred P. Sloan Foundation Fellow. Research supported in part by the National Science Foundation under Grant No. GP38627X.

†Work supported by the Stanford Synchrotron Radiation Project, the National Science Foundation, Grant No. GH39525, and the U. S. Army Research Office, Durham.

[1] K. Kawasaki, Ann. Phys. (New York) 61, 1 (1970).

[2] B. I. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969).

[3] P. C. Hohenberg, M. DeLeener, and P. Resibois, Physica (Utrecht) 65, 515 (1973).

[4] K. Kawasaki, in *Critical Phenomena, Proceedings of the International School of Physics "Enrico Fermi," Course LI*, edited by M. S. Green (Academic, New York, 1971).

[5] K. Kawasaki, J. Appl. Phys. 41, 1311 (1970).

[6] B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. Lett. 32, 1289 (1974), have chosen different streaming velocities suitable to other physical systems. We note that these streaming velocities depend on the general form of the equations of motion and not on the detailed structure of the interaction.

[7] For background see K. G. Wilson and J. Kogut, Phys. Rep. 12C, 76 (1974); S. Ma, Rev. Mod. Phys. 45, 589 (1973); B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. B 10, 139 (1974).

[8] The linear response function is defined by $G(k,\omega)=\langle S\rangle/h$ for infinitesimal external magnetic field $h\propto\exp(ik\cdot x-i\omega t)$; $\langle S\rangle$ is the average of the spin at (x,t) in the presence of h.

[9] K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).