Solution of the Three-Body Scattering Problem in Configuration Space

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The correct asymptotic behavior of the wave function deduced from the Faddeev equations for three-particle scattering is used to determine the asymptotic conditions. With these conditions, the equivalent set of partial differential equations in configuration space has a unique solution and is numerically solved for two examples. Comparison is made with results from two other basically different methods.

The Schrödinger equation for the two-particle wave function is known to have a unique solution for given asymptotic boundary conditions. For three mutually interacting particles the asymptotic behavior of the wave function was insufficiently known to determine it. Faddeev proposed a set of integral equations in momentum space to avoid this difficulty.[1] Using the Faddeev results for the momentum-space behavior of the wave function, one can deduce its correct asymptotic form in any direction of configuration space.[2,3] The boundary conditions obtained in this way make the solution of the Schrödinger equation unique.[4] The configuration space having proved to be suitable for finding the bound-state wave function in realistic cases such as ^{3}H and ^{3}He by using the Faddeev equations rather than the Schrödinger equation for purely practical reasons,[5] we are led to make the same attempts for scattering states.

The Faddeev equations expressed as partial differential equations in configuration space read

$$[(\hbar^2/m)(\Delta_{\vec{x}_i}+\Delta_{\vec{y}_i})+E-V_i(x_i)]\psi_{\vec{q}_1}{}^i(\vec{x}_i,\vec{y}_i)=V_i(x_i)\sum_{j\neq i}[\delta_{j1}\varphi_1(\vec{x}_1)\exp(i\vec{q}_1\cdot\vec{y}_1)+\psi_{\vec{q}_1}{}^j(\vec{x}_j,\vec{y}_j)], \tag{1}$$

where $\vec{x}_i$ stands for the distance between particles j and k, and $\sqrt{3}\vec{y}_i/2$ is the distance between particle i and the center of mass of the j-k pair; this pair interacts via the potential V_i, supposed here to have only one bound state, the wave function of which is $\varphi_i(\vec{x}_i)$. For simplicity we have considered here particles of equal mass m, and our Eq. (1) is specialized for the scattering of particle 1 with initial relative momentum $2\vec{q}_1/\sqrt{3}$ on the bound state of particles 2 and 3.

We seek a smooth bounded solution of the set (1) whose behavior for large hyperradius $\rho=(x_i^2+y_i^2)^{1/2}$ is

$$\psi_{\vec{q}_1}{}^i=U^i(\vec{x}_i,\vec{y}_i)+U_0{}^i(\vec{x}_i,\vec{y}_i), \tag{2}$$

where

$$U^i(\vec{x}_i,\vec{y}_i)=\varphi_i(\vec{x}_i)[A^i(\hat{y}_i)+O(y_i^{-1})]\exp(iq_iy_i)/y_i, \tag{3}$$

in which the relative momentum between free particle i and bound pair j-k is $2q_i/\sqrt{3}$, and

$$U_0{}^i(\vec{x}_i,\vec{y}_i)=[A_0{}^i(\hat{x}_i,\hat{y}_i,x_i/y_i)+O(\rho^{-1})]\exp[i(mE/\hbar^2)^{1/2}\rho]/\rho^{5/2}. \tag{4}$$

We require that the unknown amplitudes A^i and $A_0{}^i$ be smooth and bounded functions. Then one can show that the solution of the set of partial differential equations (1) with the boundary conditions (2) is unique and the same as the Fourier transform of the solution of the Faddeev integral equations.[4] It is important to note that the asymptotic form (2) is true in any direction but the first term on the right-hand side is dominant in the small-x_i region. Once the set (1) is solved, the wave function is given by

$$\psi_{\vec{q}_1}(\vec{x}_1,\vec{y}_1)=\varphi_1(\vec{x}_1)\exp(\vec{q}_1\cdot\vec{y}_1)+\sum_{i=j}^{3}\psi_{\vec{q}_1}{}^i(\vec{x}_i,\vec{y}_i). \tag{5}$$

Then the amplitudes A^i describe elastic and rearrangement processes while the breakup amplitude is

obtained from the sum of amplitudes $A_0{}^i$.

In order to illustrate the usefulness of the set of equations (1) with boundary conditions (2), we present their numerical solution for two different situations. First, we consider the S-wave scattering of a spinless particle on two identical bound particles. The potential is of Gaussian form acting in relative S state only.[6] The set (1), once symmetrized, becomes

$$[(\hbar^2/m)(\partial^2/\partial x^2+\partial^2/\partial y^2)+E-V(x)]\psi(x,y)=V(x)\int_{-1}^{1}du\,(xy/x'y')[\varphi(x')\sin(qy')+\psi(x',y')] \tag{6}$$

for the unknown $\psi(x_1,y_1)=x_1y_1\psi_{\bar{q}_1}{}^1(x_1,y_1)$, where

$$x'=\tfrac{1}{2}(x^2-2\sqrt{3}\,xyu+3y^2)^{1/2},\quad y'=\tfrac{1}{2}(y^2+2\sqrt{3}\,xyu+3x^2)^{1/2},$$

and $h^2/m=41.496$ MeV fm^2; while the asymptotic condition (2) becomes

$$\psi(x,y)=[A+O(y^{-1})]\varphi(x)\exp(iqy)+[A_0(x/y)+O(\rho^{-1})]\exp[i(mE/\hbar^2)^{1/2}\rho]/\rho^{1/2}. \tag{7}$$

The radial part of the two-particle bound state is $\varphi(x)$ and the constant amplitude A is connected with the phase shift δ and absorption coefficient η through the usual relation

$$A=(\eta e^{2i\delta}-1)/2i. \tag{8}$$

The whole process of solution is the generalization of the usual method to solve the two-body Schrödinger equation. We use here a finite-difference approximation for Eq. (6) in the variables ρ and $\theta=\arctan(y/x)$; then, starting from the origin, where the solution is regular ($\psi=0$ on both axes) a step-by-step elimination determines the real matrix of an inhomogeneous linear relation between values of ψ on two neighboring arcs of circle. The rank of this matrix is only the number of discretized θ values. This process is continued until a value of the radius ρ is obtained large enough to be in a domain where the asymptotic form (7) is true. The discretized asymptotic form (7) will lead us to another (homogeneous) linear relation between the values of ψ on the last two arcs of circle once the unknowns A and $A_0(\theta)$ are eliminated. For this, we first get rid of $A_0(\theta)$ by considering the ratio of the quantity $\psi(x,y)-A\varphi(x)\exp(iqy)$ on the two arcs; then A is eliminated by taking it as the value of $\psi(x,y)/\varphi(x)\exp(iqy)$ for x small enough so that the behavior $\varphi(x)\exp(iqy)$ is dominant.[7] The complex coefficients—when $E>0$—of this linear relation together with the real coefficients of the first linear relation enable us to find the values of ψ on one of the arcs. Finally the amplitude A is just given by the ratio $\psi(x,y)/\varphi(x)\exp(iqy)$, x small, and then the amplitude $A_0(x,y)$ is obtained by identifying the solution with its asymptotic form (7).

The numerical accuracy is checked by examining the stability of the results with respect to the discretization and the radius of matching with the asymptotic form. There are other less trivial tests: (i) The values of ψ on an arc of circle enable one to compute the solution everywhere, so that one can compute again the amplitude A by its usual integral expression; the consistency of values of A is a useful check. (ii) Below the breakup threshold—$E<0$—the absorption coefficient η should be equal to 1, providing an easy check.

In the present model computation the results become stable when the matching radius is greater than 28 fm and the agreement between differently computed values of A varies from 1% to 10% in the worst case (small $|A|$) while, for $E<0$, the absorption coefficient is found to differ by less than 0.5% from unity. We give some phase shifts and absorption coefficients in Table I. Comparison with results obtained by Revai and Raynal[6] shows an agreement within 3° for the phase shifts below threshold and within 10% for elastic cross sections above threshold. In any case, the significant figures in Table I reflect the estimated accuracy of our results.

Our second example deals with the somewhat more realistic situation of n-d scattering in an $L=0$, $J=\frac{3}{2}$ state. With the Yukawa type-III po-

TABLE I. Phase shifts and absorption coefficients for scattering of three identical particles with Gaussian interactions (Ref. 6).

E (MeV)	δ (deg)	η
−1.6	66	
−0.6	33	
4	−12	0.93
10	−34	0.75
20	−57	0.50

tential used by Kloet and Tjon,[8] which has a soft repulsive core, we find the phase shift ${}^4\delta_0 = 68°$ and absorption coefficient ${}^4\eta_0 = 0.97$ at $E_{1ab}{}^n = 14.1$ MeV, to be compared with the results ${}^4\delta_0 = 72.5°$ and ${}^4\eta_0 = 0.975$ obtained in Ref. 8. At $E_{1ab}{}^n = 46.3$ MeV we find ${}^4\delta_0 = 35°$, ${}^4\eta_0 = 0.90$, compared to 38.7° and 0.883. At $E_{1ab}{}^n = 3.27$ MeV our results were unstable, possibly because of threshold effects which will require special treatment. In view of the present state of the three-body scattering problem and the fundamental differences of the methods in Refs. 6 and 8, we find this general agreement rather promising.

Our above-described approach has the advantage of providing a direct solution to the problem, thus being free from any convergence consideration. The discretization problem which remains arises anyway with all other methods. The direct solution is made possible by the use of configuration space since the local nature of differential operators (compared with integral operators) makes the rank of matrices we have to handle an order of magnitude smaller than for the direct solution of Faddeev integral equations. Moreover the use of configuration space gets rid of any singularities usually associated with the momentum-space integral equations. The practical advantages of this method are similar to those that the Schrödinger equation has over the Lippmann-Schwinger equation. Finally we are here faced with the preliminary problem of building the two-body bound-state wave function rather than the two-body t matrix occurring in the integral equations, which is much easier for the same reasons. Taking advantage of the great simplicity and efficiency of this method, we are presently dealing with the realistic problem of three-nucleon scattering with a tensor force.

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Single-Pion Production at 100 GeV/c: A Detailed Test of Factorization

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A relatively bias-free comparison is made of inclusive and semi-inclusive charged-pion single-particle spectra in π^+p and pp interactions. Inclusive Mueller factorization is satisfied by π^- but not π^+ production. Factorization holds for proton-target fragmentation into π^- for a fixed number of π^- backward in the c.m. system, but not for fixed total multiplicity. In the central region, $\pi^+p \to \pi^\pm$ shows an $s^{-1/4}$ approach to the same value, ~0.78, seen in other reactions.

We report results of an exposure of the Fermi National Accelerator Laboratory 30-in. hydrogen bubble chamber to an unseparated but tagged π^+ and p beam at 100 GeV/c. The mixed beam allows comparison of the secondary pion spectra from high-energy π^+p and pp collisions with minimum biases, providing an excellent test of factorization: independence of secondary spectra from the identity of the incident projectile.

While factorization is an important experimental matter in itself, it is of particular interest and perhaps crucial to Mueller-Regge theory.