New Trapped-Electron Instability*

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We have found a new instability that is driven by a radial gradient of the electron temperature or by the electron drift in the "unfavorable" curvature of the magnetic field lines for high-temperature confined plasmas in which a fraction of the electron population is magnetically trapped. The relevant mode is standing along the magnetic field, is symmetric about the point of minimum field, and has relatively short wavelengths across the magnetic field.

Experiments on high-temperature magnetically confined plasmas have reached regimes where the electron collision frequencies are sufficiently low that a considerable fraction of the electron population can have magnetically trapped orbits. We point out that a new type of plasma mode can be excited in these regimes and that this may lead to a significantly enhanced electron-thermal-energy transport across the magnetic field over the values that are expected from the effects of electron-electron collisions alone.[1]

We recall that the best-known trapped-particle mode, among those which do not require that a significant fraction of the ion population be trapped, is the electron-drift mode which has a growth rate resulting from the combined effects of trapped-electron collisions and a radial temperature gradient.[2] The mode that we present here can also be driven by the radial temperature gradient of the electrons, but does not necessarily require the effects of collisions in order to be unstable,[3] and has considerably larger growth rates and shorter transverse wavelengths than the just mentioned drift mode. For selected values of the transverse wavelength the same mode is driven unstable by the drift of the ions and the trapped electrons in the unfavorable magnetic field curvature.

At first we refer, for simplicity, to a one-dimensional plane model in which we consider two electron populations in order to simulate the effects of trapped electrons that are typical of two-dimensional confinement configurations,[4] such as an axisymmetric torus. Then all equilibrium quantities depend only on x, the confining magnetic field B being in the z direction. The limit $8\pi n(T_i+T_e)/B^2 \ll 1$ is considered, n representing the particle density and $T_{e,i}$ the electron and ion temperatures. The relevant modes are electrostatic and the fluctuating electric field is $\vec{E}_1=-\nabla\tilde{\Phi}$, where $\tilde{\Phi}=\tilde{\varphi}(x)\exp(-i\omega t+ik_y y+ik_\parallel z)$. The proper range of transverse wavelengths is such that $b_i \equiv \frac{1}{2}k_y{}^2\rho_i{}^2 \gtrsim 1$, $\rho_i=v_{\text{th}i}/\Omega_i$ being the average ion gyroradius, $v_{\text{th}i}$ the ion thermal velocity, and Ω_i the ion gyrofrequency. We refer to modes localized about $x=x_0$ with $d\ln\tilde{\varphi}/dx<k_y$ so that we can take $\tilde{\varphi}(x)\approx\tilde{\varphi}(x_0)$ for our considerations. Considering the frequency range $\omega>k_\parallel v_{\text{th}i}$, the perturbed ion density[3] is, neglecting ion-temperature gradients,

$$\tilde{n}_i=-\frac{en}{T_i}\tilde{\Phi}\left\{1-\left[1-\frac{\omega_{*i}}{\omega}\right]I_0(b_i)\exp(-b_i)\left(1+\frac{\overline{\omega}_{Di}}{\omega}\right)\right\}, \tag{1}$$

where $\omega_{*i}=k_y v_{*i}$, $v_{*i}=(cT_i/eB)(d\ln n/dx)$, and $\overline{\omega}_{Di}=k_y g_i/\Omega_i$, where g_i is an effective gravitational field, acting on the ions, that is introduced in order to simulate the drift of the ions in an unfavorable magnetic curvature region.

We simulate the modes which can be excited in a two-dimensional equilibrium configuration, in which the magnetic field is periodically modulated by a value ΔB over a distance L, by taking $k_\parallel=2\pi/L$. Thus the average transit frequency $\langle\omega_t\rangle$ of circulating particles is of order $2\pi v_{\text{th}}/L$ and the average bounce frequency $\langle\omega_b\rangle$ of trapped particles is of order $(2\pi v_{\text{th}}/L)(\Delta B/B)^{1/2}$. Notice that for frequecies $\omega>\langle\omega_t\rangle_i$, as we have considered, the expression (1) for $\tilde{n}_i$ is valid regardless of whether a finite fraction of the ion population is trapped.

We evaluate $\tilde{n}_e$ in the limit $\omega<\langle\omega_b\rangle_e$ and model the effects of trapped electrons, as indicated in Ref. 4, by assuming that in a slab geometry they constitute a separate cold population. Then their perturbed density is simply given by

$$-i\omega\tilde{n}_{eT}+i\overline{\omega}_{De}\tilde{n}_{eT}-ik_y(c/B)\tilde{\Phi}\,dn_{eT}/dx=0, \tag{2}$$

while for circulating electrons

$$\tilde{n}_{eC} = (e/T_e)\tilde{\Phi}\, n_{eC}\,, \tag{3}$$

where $n_{eC}+n_{eT}=n$, $dn_{eT}/dx=(n_{eT}/n)dn/dx$, $\overline{\omega}_{De}=k_y g_e/\Omega_e$, and g_e is an effective gravitational field, acting on the trapped electrons. Notice that $\overline{\omega}_{De}=-(T_e/T_i)\overline{\omega}_{Di}\sim cT_e/(eBR_0)$, where R_0 is the typical radius of magnetic curvature. Thus $R_0>0$ corresponds to a magnetic curvature that is in the opposite direction of the density gradient. The perturbed quasineutrality condition $\tilde{n}_i=\tilde{n}_{eC}+\tilde{n}_{eT}$ leads to the dispersion relation

$$\left(1-I_0(b_i)\exp(-b_i)+\frac{n_{eC}}{n}\frac{T_i}{T_e}\right)-\frac{\omega_{*i}}{\omega}\left(\frac{n_{eT}}{n}-I_0(b_i)\exp(-b_i)\right)-\frac{\omega_{*i}}{\omega^2}\left(\overline{\omega}_{De}\frac{n_{eT}}{n}-\overline{\omega}_{Di}I_0(b_i)\exp(-b_i)\right)=0. \tag{4}$$

Then we see that when $I_0(b_i)\exp(-b_i)\ll n_{eT}/n$, corresponding to $b_i>(n/n_{eT})^2$, a mode with phase velocity transverse to the magnetic field in the direction of the ion diamagnetic velocity and a frequency proportional to the concentration of trapped electrons is obtained. When $b_i<1$ the ordinary electron-drift mode[2] is obtained. When $b_i>1$ is such that $n_{eT}/n\simeq I_0(b_i)\exp(-b_i)$, the last term in Eq. (4), due to magnetic curvature, produces an instability which is fluidlike, in that it does not depend on a mode-particle resonance involving only a small fraction of the electron distribution.

A more complete dispersion relation for this mode is obtained by adding to the expression for $\tilde{n}_e$ two terms of higher order in $\omega/\langle\omega_b\rangle_e$, and a term due to trapped-electron collisions. Thus, according to the results we present in the following,

$$\tilde{n}_e=\tilde{n}_{eT}+\tilde{n}_{eC}=e\tilde{\Phi}\frac{n}{T_e}\left\{1-\frac{n_{eT}}{n}\frac{\omega_{*e}}{\omega}\left[\frac{\omega^2}{\langle\omega_b\rangle_e^2}(1-\eta_e)\alpha_2-(1+\eta_e)\frac{\overline{\omega}_{De}}{\omega}\alpha_4-\alpha_1 + i(1-\tfrac{3}{2}\eta_e)\left(\frac{\omega^3}{\langle\omega_b\rangle_e^3}\alpha_3+\frac{\nu_{\text{eff}}^{1/2}}{\langle\omega_b\rangle_e^{3/2}}\omega\alpha_5+\frac{\nu_{\text{eff}}}{\omega}\alpha_6\right)\right]\right\}, \tag{5}$$

where $\omega_{*e}=-\omega_{*i}T_e/T_i$, and $r_n=-(d\ln n/dx)^{-1}$. The quantities α_1, α_2, α_3, α_4, α_5, and α_6 are finite coefficients and $\eta_e\equiv(d\ln T_e/dx)/(d\ln n/dx)$. In evaluating the effects of collisions we have adopted a simplified collision operator, the type used in Ref. 2, in which $\nu=\nu_{\text{eff}}(T_e/\epsilon)^{3/2}$ represents the collision frequency, $\epsilon=\frac{1}{2}m_e(v_\perp^2+v_\parallel^2)$ is the electron kinetic energy, and $\nu_{\text{eff}}\sim(B/\Delta B)\nu_e$, ν_e being the average electron collision frequency,[1] and we have assumed $\nu_{\text{eff}}<\omega$. We also recall that, in an equilibrium configuration in which the magnetic field is periodically varying over a connection length L, the mode-particle resonances involving trapped electrons are of the form[4] $\omega=\omega_b(\epsilon,\mu)$, where $\mu=m_e v_\perp^2/2B$ is the magnetic moment. The contribution of this resonance is represented by the α_3 term. The complete model of the dispersion equation, including the ion-temperature gradient, is

$$\left(1-I_0(b_i)\exp(-b_i)+\frac{n_{eC}}{n}\frac{T_i}{T_e}\right)+\frac{\omega_{*i}}{\omega}\{I_0(b_i)-\eta_i[I_0(b_i)-I_1(b_i)]b_i\}\exp(-b_i)\left(1+\alpha_7\frac{\overline{\omega}_{Di}}{\omega}\right)$$
$$=\frac{n_{eT}}{n}\frac{\omega_{*i}}{\omega}\left\{\alpha_1+(1+\eta_e)\frac{\overline{\omega}_{De}}{\omega}\alpha_4-\frac{\omega^2}{\langle\omega_b\rangle_e^2}(1-\eta_e)\alpha_2 - i(1-\tfrac{3}{2}\eta_e)\left[\frac{\omega^3}{\langle\omega_b\rangle_e^3}\alpha_3+\left(\frac{\nu_{\text{eff}}}{\langle\omega_b\rangle_e}\right)^{1/2}\frac{\omega}{\langle\omega_b\rangle_e}\alpha_5+\frac{\nu_{\text{eff}}}{\omega}\alpha_6\right]\right\}. \tag{6}$$

We see that a nonresonant instability is produced by the magnetic-curvature term when

$$\{I_0(b_i)-\eta_i[I_0(b_i)-I_1(b_i)]b_i\}\exp(-b_i)\simeq(n_{eT}/n)\alpha_1\,,$$

corresponding to $b_i\sim B/(\Delta B)$, where $\eta_i=(d\ln T_i/dr)/(d\ln n/dr)$, or when $b_i>B/(\Delta B)$ and $\eta_e\gtrsim n_{eT}R_0T_i/nr_nT_e$. A nonresonant mode can also be produced by the term in α_2 for $\eta_e\gtrsim\langle\omega_b\rangle_e^2/\omega^2$ and in these cases we expect it to manifest itself with more macroscopic effects than when it depends on a wave-particle resonance, such as for $\frac{2}{3}<\eta_e\sim1$ and $b_i>B/(\Delta B)$. However, for $\frac{2}{3}<\eta_e$, $b_i>B/(\Delta B)$, and $\omega^3/\langle\omega_b\rangle_e^3\leq\nu_{\text{eff}}/\omega<1$, the relevant growth rate is collisional and still nonresonant. For larger values of the collision frequency, that is $\omega<\nu_{\text{eff}}<\langle\omega_b\rangle_e$, the collisional term replacing the last one in Eq. (6) is $-i\alpha_8(1+3\eta_e/2)\omega/\nu_{\text{eff}}$ and it is sufficient to damp the considered resonant mode. Now it is easy to verify that the condition $\nu_{\text{eff}}/\omega<1$ is well satisfied in a number of significant experiments. For instance, if we consider electron temperatures $T_e \gtrsim 2$ keV with plasma densities of about 2×10^{13} cm^{-3} as obtained in toroidal devices with inverse aspect ratios $\epsilon_0 \simeq 0.2$, we evaluate the ratio $\nu_{\rm eff}/\langle\omega_b\rangle_e$ in a range around 10^{-2}; there is an ample interval for ω corresponding to growing modes. We also notice that the interval of $\nu_{\rm eff}/\omega$ within which we find instability is just the complement of the one for which the electron-drift mode[2] is unstable.

We can estimate the evolution of the average (nonfluctuating) electron distribution function by the quasilinear approximation, in the limit where $\eta_e \sim 1$ and $b_i \gg 1$; the electron thermal transport across the field is considerably larger than the particle transport, as the density fluctuation for both species is in phase with the electric potential fluctuation [see Eq. (1)]. When b_i is such that the fluidlike instability, driven by the effect of unfavorable magnetic curvature on the ions and the trapped electrons, occurs as indicated by Eq. (4), the particle transport across the magnetic field also becomes important, but the use of quasilinear theory to estimate this transport is hardly justifiable.

Finally, we give an outline of the theory that it is necessary to develop in order to obtain the just mentioned results, including the effects of magnetic shear as well as those of the geometry of the considered confinement configuration. Here ζ and θ denote, respectively, the toroidal and poloidal angular coordinates; the magnetic field can be represented by $\vec{B} \approx [B_0\vec{e}_\zeta + B_\theta{}^0(r)\vec{e}_\theta]/[1+(r/R_0)\cos\theta]$. The field rotational transform is given by $1/q \approx R_0 B_\theta/rB_\zeta$ in the large-aspect-ratio limit $1/\epsilon_0 \equiv R_0/r \gg 1$ that we consider, so that $L \approx 2\pi R_0 q$ and $\Delta B/B \approx 2\epsilon_0$. The appropriate electric potential in the neighborhood of a rational surface $r = r_0$ can be written, for convenience, as

$$\tilde{\Phi} = \tilde{\varphi}_m(\theta)\psi(r-r_0)\exp\{-i\omega t + in^0[\zeta - q(r)\theta] + iS(r)F(\theta)\},$$

where $\tilde{\varphi}_m(\theta)$ is periodic in θ, with period 2π, while $F(\theta)$ is monotonic in θ so that $F(\theta) = \int_0^\theta G(\theta')d\theta'$, $G(\theta)$ being also periodic and even around $\theta = 0$, with $F(\theta = 2\pi) = 2\pi$. The surface $r = r_0$ is such that $q(r_0) = m^0/n^0 \equiv q_0$, m^0 and n^0 being integers, and we have defined the radial variable $S(r) = n^0[q(r) - q_0]$. Thus $\tilde{\varphi}_m(\theta)$ represents the mode amplitude modulation along a given magnetic field line for $r = r_0$.

The mode of interest is of *standing* type along the magnetic field lines and is characterized by $\tilde{\varphi}_m(\theta)$ *even* around $\theta = 0$ with its maximum at $\theta = 0$ and $\tilde{\varphi}_m(\theta = \pm\pi) = 0$. In this case we may choose $G(\theta) \approx 2\pi \times \delta(\theta \pm \pi)$. The function $\psi(r - r_0)$ is considered to be localized over a distance $\Delta r < r_0$, which we take to be related to the scale distances $r_n \sim r_T \equiv -(d\ln T_e/dr)^{-1}$. Since we consider relatively large values of m^0, such that $\Delta r_s \equiv |n^0 dq/dr|^{-1} \ll r_n \sim r_T$, $\psi(r - r_0)$ can be taken as a nearly periodic function over $\Delta r_s < \Delta r$ as can be verified *a posteriori*. Thus we refer to $\psi(S)$ in the interval $-\frac{1}{2} \leq S \leq \frac{1}{2}$ and notice that for the mode of interest $(\Delta r_s)^2 > \rho_{be}{}^2$, ρ_{be} being the average width of a trapped-electron banana orbit ($\rho_{be} \sim \epsilon_0{}^{1/2}\rho_e B/B_\theta$).

The equilibrium electron distribution is assumed to be of the form $f_e = f_{Me}(1 + \hat{f}_e)$, where

$$f_{Me} = [n_e/(2\pi T_e/m_e)^{3/2}]\exp(-\epsilon/T_e),$$

$$\hat{f}_e = (v_\zeta/|\Omega_{\theta e}|)[d\ln n_e/dr - (d\ln T_e/dr)(\tfrac{3}{2} - \epsilon/T_e)],$$

and

$$\Omega_{\theta e} = eB_\theta/m_e c.$$

The perturbed electron density, when we neglect collisions, is obtained by integration of the linearized Vlasov equation along unperturbed electron orbits for which we use the guiding-center approximation. Thus, circulating orbits correspond to $0 \leq \Lambda < 1 - r/R_0$ and trapped-particle orbits to $1 - r/R_0 \leq \Lambda \leq 1 + r/R_0$, for $\Lambda = \mu B_0/\epsilon$. It is convenient to decompose $\tilde{\varphi}_m(\theta)$ in harmonics of the particle-orbit periodicity frequency so that

$$\tilde{\varphi}_m(\theta)\exp\{iS[F(\theta) - 2\omega_t\hat{t}]\} = \textstyle\sum_p \tilde{\Phi}^{(p)}(\Lambda, S)\exp(i2p\omega_t\hat{t}),$$

referring to circulating particles for which $\hat{t} = \int^\theta d\theta/\dot{\theta}$ is a parametric function of θ, $\omega_t = 2\pi/\tau_t$, $\tau_t = 2\int_{-\pi}^{\pi} d\theta/\dot{\theta}$, and $\dot{\theta} \approx \dot{\zeta}/q \approx v_\parallel/R_0 q$, so that the transit frequency $\omega_t = \omega_t(\epsilon, \Lambda)$. For trapped particles we consider instead $\tilde{\varphi}_m(\theta) = \sum_p \tilde{\Phi}^{(p)}(\Lambda)\exp(ip\omega_b\hat{t})$, where $\hat{t} = \int^\theta d\theta/\dot{\theta}$, $\omega_b = 2\pi/\tau_b$, $\tau_b = 2\int_{-\theta_0}^{\theta_0} d\theta/|\dot{\theta}|$, and $\pm\theta_0$ correspond to the orbit turning points. In particular, $\langle\omega_b\rangle_{e,i} = (v_{{\rm th}e,i}/2qR_0)(2r/R_0)^{1/2}$ and we refer to the frequency range $\langle\omega_b\rangle_i < \omega \lesssim \langle\omega_b\rangle_e$. Then $\tilde{n}_i$ is still given by (1) with the addition of terms due to η_i as indicated in (6) and the replacement of the $\overline{\omega}_{Di}$ term by one with different b_i dependence. Then we derive the following quadratic form:

$$\langle\!\langle \tilde{n}_i \tilde{\Phi}^* \rangle\!\rangle \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_{-1/2}^{1/2} dS \tilde{n}_i \tilde{\Phi}^* = \langle\!\langle \tilde{n}_e \tilde{\Phi}^* \rangle\!\rangle \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_{-1/2}^{1/2} dS \tilde{n}_e \tilde{\Phi}^*,$$

and expand it in the limit $\omega < \langle \omega_b \rangle_e$. Then we reproduce (6) with α_5, α_6, and α_8 given as ratios of integrals $\hat{\Pi}$, given in Ref. 3, which are finite, positive definite, and quadratic in $|\tilde{\Phi}^{(p)}(\Lambda, S)|$. In this case ω_{De} is replaced by a function $\omega_{De}^{(0)}$ that is the time-average magnetic-curvature drift frequency $\omega_{De}^{(0)} = \tau_b^{-1} \oint d\hat{t}' \omega_{De}(\hat{t}')$ seen by the trapped electrons, where $\omega_{De} = (m^0/r_0) v_D$ and v_D is the poloidal component of the magnetic-curvature drift velocity. The frequency $\omega_{De}^{(0)}$ is of order $\overline{\omega}_{De}$. In addition, $\overline{\omega}_{Di} I_0(b_i)\exp(-b_i)$ in (1) is replaced by $\overline{\omega}_{Di}\gamma_7(b_i, \eta_i)\cos\theta$, where

$$\gamma_7 = \exp(-b_i)\{2I_0 - b_i(I_0 - I_1) - \eta_i[(b_i - 2)I_0 + (3b_i - 2b_i^2)(I_0 - I_1)]\},$$

and the argument of the Bessel functions is b_i. In this context, $\overline{\omega}_{Di}$ represents the effect of the favorable curvature that is seen by all ions, circulating and trapped, in the region where the mode amplitude is a maximum.

Notice that in the cases where two terms such as the one on the left-hand side and the first one on the right-hand side are the largest in Eq. (6), we can extract variational forms from it which can be used to estimate the corresponding values of ω and to gain information, from the features of the extremizing functions $\tilde{\varphi}_m(\theta)$ and $\psi(S)$, on the relevant eigenfunctions. This is the sense in which (6) is the model dispersion equation for the present problem. In particular we see that whenever the terms in α_2 and α_3 do not contribute to the lowest-order form of (6), the contribution of circulating particles disappears from it and we can take $\psi(S) \simeq 1$. If we evaluate the effects of collisions as indicated earlier, we can identify the coefficients α_5, α_6, and α_8 in terms of Π integrals that are given in Ref. 3. We have not found any appreciable stabilizing effect of magnetic shear from the considered quadratic form.

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