176, 1456 (1968). In our work we are more explicit about the shape and time behavior of the average radiated intensity, and on the condition for superfluorescence (16).

[16]N. Skribanowitz *et al.*, Phys. Rev. Lett. 30, 309 (1973).

ϵ-Expansion Solution of Wilson's Incomplete-Integration Renormalization-Group Equations

Prabodh Shukla and Melville S. Green
Hebrew University, Jerusalem, Israel, and Temple University, Philadelphia, Pennsylvania 19122
(Received 23 May 1974)

Wilson's incomplete-integration renormalization-group equations have been solved in $4-\epsilon$ dimensions for an arbitrary cutoff function. The two relevent exponents are computed to order ϵ and the exponent η is computed to order ϵ^2. To order ϵ the exponents agree with the sharp-cutoff renormalization group. In order ϵ^2, however, η appears to depend on the choice of the cutoff function.

One of the most beautiful but puzzling features of the renormalization-group approach is the fact that apparently *ad libitum* changes in the parameters of the physical system and the renormalization group itself do not affect the results of the theory. This of course is the theoretical basis of the physical fact of the universality of critical phenomena in a wide variety of systems and substances. Invariance of exponents and other quantities with respect to changes in parameters of the system have found their explanation in the concepts of relevant and irrelevant observables introduced by Kadanoff.[1] The effect of changes in the renormalization group itself are less well understood. Wegner[2] and Jona-Lasinio[3] have given a very general definition of a renormalization group and have given conditions under which two renormalization groups give the same exponents. DiCastro[4] has shown in the context of the ϵ expansions that the renormalization group (RNG) of Gell-Mann and Low and the renormalization group of Wilson (sharp cutoff) yield the same exponents. It is the purpose of this Letter to carry out the ϵ expansion for yet another renormalization group, Wilson's incomplete-integration RNG.[5]

We start with Wilson's incomplete-integration RNG equation,

$$\frac{\partial H}{\partial t}=\int_k\left(\frac{d}{2}\sigma_k+k\cdot\nabla_k\sigma_k\right)\frac{\delta H}{\delta\sigma_k}+\int_k[b+\beta(k)]\left(\frac{\delta H}{\delta\sigma_k}\frac{\delta H}{\delta\sigma_{-k}}+\frac{\delta^2 H}{\delta\sigma_k\delta\sigma_{-k}}+\sigma_k\frac{\delta H}{\delta\sigma_k}\right),$$

where b is a constant and $\beta(k)$ is any positive increasing analytic function of k^2 with $\beta(0)=0$. The effective cutoff on the wave vector is e^{-t} and $\beta(k)$ determines the shape of this cutoff. σ_k is the Fourier component of the spin field corresponding to wave vector k. We attempt to determine the fixed-point Hamiltonian H^* such that

$$\partial H/\partial t|_{H=H^*}=0,$$

and we make the expansion

$$H^*=-\tfrac{1}{2}\int_k U_2^*(k)\sigma_k\sigma_{-k}-\tfrac{1}{4!}\int_k\int_{k1}\int_{k2}U_4^*(k_1k_1,k_2,k_3)\sigma_k\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}-\cdots.$$

We fix $\beta(k)$ and vary b so as to determine U_{2n}^* which are analytic. We make the following *Ansatz* for b and U_{2n}^*:

$$b=1+b_1\epsilon+b_2\epsilon^2+\ldots;\quad U_2^*=U_{20}^*+U_{21}^*\epsilon+U_{22}^*\epsilon^2+\ldots;\quad U_4^*=U_{41}^*\epsilon+U_{42}^*\epsilon^2+\ldots;\quad U_6^*=U_{62}^*\epsilon^2+\cdots.$$

We skip the explicit writing of the arguments of the U_{2n}^* when it is not difficult to see what they should be. We have chosen $b_0=1$ in order that $U_{20}^*(k)$ be analytic with an expansion that begins with k^2.

The above expansion enables us to truncate the RNG equations. To order ϵ^2, we have

$$k\cdot\nabla_k U_{20}^*=2[1+\beta(k)](1-U_{20}^*)U_{20}^*;\quad k\cdot\nabla_k U_{21}^*=2[1+P(k)]U_{21}^*+2b_1(1-U_{20}^*)U_{20}^*+\int_{k_1}[1+\beta(k_1)]U_{41}^*;$$

$$k\cdot\nabla_k U_{22}^*=2[1+P(k)]U_{22}^*-2[1+\beta(k)](U_{21}^*)^2+2b_1(1-2U_{20}^*)U_{21}^*+2b_2(1-U_{20}^*)U_{20}^*$$
$$+\int_{k_1}[1+\beta(k_1)]U_{42}^*+b_1\int_{k_1}[1+\beta(k_1)]U_{41}^*+\int^{\epsilon}[1+\beta(k_1)]U_{41}^*;$$

$$\sum_i k_i \cdot \nabla k_i U_{41}{}^* = \sum_i P(k_i) U_{41}{}^*; \quad \sum_i k_i \cdot \nabla_{k_i} U_{42}{}^* = \sum_i P(k_i) U_{42}{}^* + U_{41}{}^* + \sum_i \{ b_1 (1 - 2U_{20}{}^*) - [1 + \beta(k_i)] U_{21}{}^* \} U_{41}{}^* + \int_{k_4} [1 + \beta(k_4)] U_{62}{}^*;$$

$$\sum_i k_i \cdot \nabla_{k_i} U_{62}{}^* = -2U_{62}{}^* + \sum_i P(k_i) U_{62}{}^* - 2\sum_P [1 + \beta(k + k_1 + k_2)] U_{41}{}^*(k, k_1, k_2, -k - k_1 - k_2) \times U_{41}{}^*(k_3, k_4, k_5, -k_3 - k_4 - k_5),$$

where $P(k) = \beta(k)[1 - 2U_{20}{}^*(k)] - 2U_{20}{}^*(k)$. The integrals are four-dimensional. $\int^{\epsilon}$ denotes the term of order ϵ in the expansion of $\int_k$ around dimensionality four.

We first determine $U_{20}{}^*$ and $U_{41}{}^*$ since these equations do not involve any unknown functions. Wilson has solved the nonlinear equation for $U_{20}{}^*$:

$$U_{20}{}^*(k) = ak^2 / \{ ak^2 + \exp[-2\int_0^k k^{-1} \beta(k) dk] \},$$

where a is an arbitrary constant which Wilson chooses to be 1. The equation for $U_{41}{}^*$ is a separable linear homogeneous equation whose solution is

$$U_{41}{}^* = A \prod_i f(k_i),$$

where A is a constant and $f(k) = \exp \int_0^k k^{-1} P(k)\, dk$.

With $U_{20}{}^*$ and $U_{41}{}^*$ known, we can now solve the equation for $U_{21}{}^*$ at which point we will have completely determined the fixed point to order ϵ. The equation for $U_{21}{}^*$ is a linear ordinary differential equation in which the coefficient of the first derivative vanishes when $k = 0$. This equation will have a solution analytic in k at $k = 0$ only if $b_1 = 0$. Thus $b = 1$ to order ϵ. The general analytic solution for $U_{21}{}^*$ is

$$U_{21}{}^*(k) = (-\tfrac{1}{2}AB + C_1 k^2) f^2(k),$$

where $B = \int_k \psi(k)$, $\psi(k) = [1 + \beta(k)] f^2(k)$, and C_1 is an arbitrary constant. The equations for $U_{62}{}^*$ and $U_{42}{}^*$ are linear inhomogeneous partial differential equations which may be solved by the method of characteristics. $U_{62}{}^*$ is determined uniquely by the requirement of analyticity at the origin,

$$U_{62}{}^* = -2A^2 \prod_i f(k_i) \sum_P g(k + k_1 + k_2),$$

where

$$g(k) = \int_0^1 \lambda\, d\lambda\, \psi(\lambda k)$$

and $\sum_P$ denotes a sum over ten terms which come from inequivalent choices of three out of the six vectors k, k_1, k_2, k_3, k_4, and k_5. In order that $U_{42}{}^*$ be analytic at the origin it is necessary that

$$A = [12 \int_k \psi(k) g(k)]^{-1}.$$

Finally b_2 is determined by the condition that $U_{22}{}^*(k)$ is analytic at $k = 0$. This yields

$$-2b_2 = -\frac{4A^2}{a} \int_{k_2} \psi(k_2) \int_{k_4} \psi(k_4) \int_0^1 \lambda\, d\lambda \frac{1}{F} \frac{d\psi(F)}{dF},$$

where

$$F = |\lambda k_2 + k_4|.$$

To determine the exponents we consider the behavior of small perturbations in the neighborhood of the fixed point. We set

$$H = H^* + O_m(\sigma) \exp(-d_m t)$$

in the RNG equation and retain only terms linear in the perturbation $O_m(\sigma)$. We write

$$O_m(\sigma) = \int_{k_1} V_1(k_1) \sigma_{k_1} + \tfrac{1}{2} \int_{k_1} \int_{k_2} V_2(k_1, k_2) \sigma_{k_1} \sigma_{k_2} + \ldots,$$

and seek values of d_m for which the V's are analytic. The linearized RNG equations for the functions V fall into two disjoint classes, one containing odd V's and one containing even V's. The exponent η is determined from the eigenvector of the odd equations which approaches the eigenvector of the Gaussian fixed point

$$V_1(k) = f(k), \quad V_3 = V_5 = \ldots = 0$$

as $\epsilon \to 0$. For this eigenvector d_m equals $\frac{1}{2}(d - 2 + \eta)$. Introducing the ϵ expansion

$$d_m = 1 - \tfrac{1}{2}\epsilon + \tfrac{1}{2}\eta, \quad \eta = \eta_0 + \eta_1 \epsilon + \eta_2 \epsilon^2 + \ldots,$$

$$V_1 = V_{10} + V_{11}\epsilon + V_{12}\epsilon^2 + \ldots,$$

$$V_3 = V_{31}\epsilon + V_{32}\epsilon^2 + \ldots; \quad V_5 = V_{52}\epsilon^2 + \ldots,$$

we obtain six equations for V_{10} to V_{52}, which are very similar to the six equations for $U_{20}{}^*$ to $U_{62}{}^*$. They may be solved by the same methods as for the equations for the U_{2n}'s. At each order η_0, η_1, and η_2 are chosen so as to make the solutions analytic. To order ϵ the results are identical to those obtained for the sharp-cutoff RNG[6]; i.e. $\eta_0 = 0$, $\eta_1 = 0$. η_2 is found to be $-2b_2$. It is relatively easy to determine the second relevant anomalous dimension to order ϵ. This is the eigenvalue corresponding to the eigenvector which approaches the Gaussian eigenvector

$$V_2 = f(k_1) f(k_2), \quad V_4 = V_6 = \ldots = 0 \text{ as } \epsilon \to 0.$$

We find, in agreement with the sharp-cutoff RNG, to order ϵ, $d_m = 2 - \frac{2}{3}\epsilon$. Thus to order ϵ the two relevant thermodynamic exponents are identical to the sharp-cutoff RNG. To order ϵ^2, however, the value of η explicitly contains $\beta(k)$ and a which characterizes $U_{20}{}^*(k)$. An important question is whether the dependence on $\beta(k)$ and a is only apparent and that through hidden identities this dependence disappears. In this connection we note that the expression is invariant under the simultaneous transformation $a \to \lambda^2 a$, $\beta(k) \to \beta(\lambda k)$. However, we have not yet resolved the question of whether the dependence on $\beta(k)$ and a is apparent or real.

The authors wish to thank the Racah Institute of Physics for generous hospitality during difficult circumstances. M.S.G. gratefully acknowledges the support of the Guggenheim Foundation, and P.S., a Temple University Fellowship.

[1]L. P. Kadanoff, in *Critical Phenomena, Proceedings of the International School of Physics "Enrico Fermi," Course LI,* edited by M. S. Green (Academic, New York, 1971).

[2]F. Wegner, talk at Conference on Renormalization Group in Critical Phenomena and Quantum Field Theory, Philadelphia, Pennsylvania, May 1973 (unpublished).

[3]G. Jona-Lasinio, in *Collective Properties of Physical Systems,* edited by B. Lundqvist and S. Lundqvist, Nobel Symposium, June 1973 (Nobel Foundation, Stockholm, 1973, and Academic, New York, 1973).

[4]C. DiCastro, Lett. Nuovo Cimento 5, 69 (1972).

[5]K. G. Wilson, as reported in K. G. Wilson and J. Kogut, Phys. Rep. 12C, 76 (1974).

[6]F. Wegner and A. Houghton, Phys. Rev. A 8, 401 (1973).

Experimental Evaluation of the Scaling Function for the Scattered Intensity in the Critical Region*

J. S. Lin and Paul W. Schmidt

Physics Department, University of Missouri, Columbia, Missouri 65201

(Recieved 9 October 1974)

A new method has been developed for analyzing small-angle x-ray scattering data from fluids in the critical region without assuming the form of the angular distribution of the scattered intensity. By use of data obtained previously, the scaling function for the scattered intensity has been evaluated for argon. This function expresses the angular distribution of the scattering in terms of $q\xi$, where ξ is the long-range correlation length, $q = 4\pi\lambda^{-1}\sin(a/2)$, λ is the x-ray wavelength, and a is the scattering angle.

When a fluid is near its critical point, correlations between density fluctuations extend over distances many times larger than atomic dimensions. These correlations can be characterized by the long-range correlation length ξ, which becomes large near the critical point. The long-range density fluctuations scatter x rays at small angles, and so, when a fluid approaches its critical point, there is a large increase in the x-ray scattering at angles no greater than a few degrees.

X-ray scattering can be used to study many of the equilibrium properties of fluids in the critical region.[1-5] The long-range correlation length ξ can be calculated from the angular dependence of the scattering, and the zero-angle scattered intensity $I(0)$ is proportional to the quantity $\rho^2 K_T$, where ρ is the number of molecules per unit volume and K_T is the isothermal compressibility. In addition,[3,5] measurements of the x-ray absorption permit an independent determination of ρ.

Since the long-range correlation length determines the angular dependence of the scattered intensity in the critical region, measurements of the scattering from a fluid near its critical point can be used to study the behavior of ξ. However, most of the methods for quantitative interpretation of the scattering data require some knowledge of the relationship between ξ and the scattered intensity, and so approximate theoretical expressions for the scattering are normally used to help interpret the scattering data.

Since the values of ξ and $I(0)$ computed from the scattering curves can often be affected by the form of the theoretical approximation selected for analysis of the data, a method for interpreting the scattering without using an approximate equation would be preferable. We have recently