netic tuning of the superconducting energy gap, E_c and k_c as a function of pressure can be extracted. The existence of the 3pp and a well-defined cutoff energy imply an upward curvature to the excitation spectrum. In addition, from pulsed time-of-flight measurements the group velocity of these phonons at k_c can be measured and comparisons with existing models for the excitation spectrum can be made. These detailed analyses and comparisons will be the subject of a future publication.[10]

In summary we have direct evidence for the existence of the 3pp operative in an energy range up to a cutoff E_c in the excitation spectrum of He II. This implies that upward dispersion does exist. Our previous conclusions from pulsed measuremens were erroneous as we were detecting phonons of energy $E > E_c$. From detailed measurements of this type we can directly compare results with various $E(k)$ models for He II.[10]

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Instability of a Vortex Array in He II*

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We propose an instability of a vortex array in the presence of axial normal-fluid flow in He II. This instability may be responsible for the influence of thermal counterflow on ion-trapping cross sections reported by Cheng, Cromar, and Donnelly.

It is generally accepted that, upon rotation, He II is threaded by an array of quantized vortex lines parallel to the axis of rotation.[1] This array exhibits oscillation modes which in the long-wavelength limit are similar to classical inertial waves and in the short-wavelength limit are related to the waves of isolated vortex lines. Hall[2] predicted these modes on the basis of a set of phenomenological two-fluid equations and was able to observe the modes and, from their dispersion relation, to determine the vortex core parameter. In this paper we use the Hall[3] equations to investigate the normal modes of a vortex array in the presence of axial normal-fluid flow. We find that some of the modes become unstable when the velocity of the axial flow exceeds some (rather small) critical value. This implied disruption of the vortex array may be the explanation for the anomalous ion-trapping results reported by Cheng, Cromar, and Donnelly[4] and may be relevant for an interpretation of some results of Williams and Packard.[5]

We begin by considering an apparently unrelated situation, that of a vortex ring in counterflow.[6] Viewed in the frame of reference in which the superfluid is at rest at infinity, a ring having radius R travels with velocity

$$V = (\kappa/4\pi R)\ln(8R/a - \tfrac{1}{2}), \quad \kappa = h/m, \tag{1}$$

where h is Planck's constant, m is the mass of a helium atom, and a is the core parameter (~1 Å). If the normal fluid is at rest in this frame, dragging of the vortex ring through the normal fluid causes the ring to lose energy steadily, and hence to decrease in size. If, on the other hand, the normal fluid is moving faster than (and in the same direction as) the ring, it is clear that energy is being added to the ring and, because of the peculiar dynamics of a ring, it will continue to grow. Equation (1) then represents the critical velocity of counterflow for which a ring of radius R becomes unstable for unlimited growth.

Simple models and several experiments suggest[7] that vortex lines will not interact with normal-fluid flow parallel to the lines. However, if a vortex line is distorted into a helix, it is apparent that normal flow along the unperturbed vortex line is not accurately parallel to the line anywhere. Such a helical distortion is a traveling distrubance or normal mode of the line, whose velocity depends only on the wavelength of the disturbance, and not on its amplitude (for small amplitudes). A vortex wave of small amplitude is certainly not to be considered a small vortex-ring-like disturbance, but the mechanism for energy transfer from the normal flow to the vortex line is similar. If the axial velocity of normal flow is larger than the velocity of some vortex wave, that wave's amplitude will grow indefinitely, until limited by some other factor. As we shall see, the phase velocity of vortex waves has a minimum value, giving rise to a critical axial normal-fluid velocity equal to that minimum value.

The simplest vortex-wave modes to picture are those for which the wave vector is along the rotation axis. These modes are circularly polarized waves in which each vortex-line element executes circular motion (opposite in sense to the rotation) in a plane perpendicular to the axis of rotation, and the dispersion relation is simply[3]

$$\omega = 2\Omega + \nu k^2. \tag{2}$$

Here $\nu = (\kappa/4\pi)\ln(b/a)$, where b is a length of the order of the vortex-line spacing, and Ω is the rotation frequency. This dispersion exhibits a Landau-like critical velocity

$$u_c = (\omega/k)_{\min} = 2(2\Omega\nu)^{1/2}, \tag{3}$$

at a critical wave number

$$k_c = (2\Omega/\nu)^{1/2}, \tag{4}$$

and at a frequency

$$\omega_c = 4\Omega. \tag{5}$$

If, instead of Eq. (2), we use an expression derived by Raja Gopal,[8] the critical velocity is reduced by about 10%.

A more detailed analysis is necessary if the energy transfer is to be explicitly considered. We take the equation of motion of the superfluid, suggested by Hall,[3] in a coordinate system rotating at frequency Ω:

$$D\vec{V}_s/Dt = \nabla\varphi + 2\vec{V}_s\times\Omega + \alpha\hat{\lambda}\times[\vec{\lambda}\times(\vec{V}_s-\vec{V}_n)] + \beta\vec{\lambda}\times(\vec{V}_s-\vec{V}_n) - \alpha\nu\hat{\lambda}\times(\vec{\lambda}\cdot\nabla)\hat{\lambda} + \nu(1-\beta)(\vec{\lambda}\cdot\nabla)\hat{\lambda}, \tag{6}$$

where $\vec{V}_s$ and $\vec{V}_n$ are the superfluid and normal-fluid velocities in the rotating coordinate system (averaged in the sense discussed by Hall), $\vec{\lambda} = \nabla\times\vec{V}_s + 2\vec{\Omega}$, $\hat{\lambda}$ is a unit vector along $\vec{\lambda}$, α and β are proportional to the mutual friction coefficients, and φ is a collection of scalar terms. $\vec{V}_n$, ρ_s, and ρ_n are taken to be constants in the modes we wish to consider, so that the equation of continuity is simply

$$\nabla\cdot\vec{V}_s = 0. \tag{7}$$

The assumption that $\vec{V}_n$ is constant is probably justified in the experimental situation as discussed by Andronikashvili *et al.*[9] Assuming that $\vec{\Omega} = \Omega\vec{e}_z$ and $\vec{V}_n = u\vec{e}_z$, we linearize Eqs. (2) and (3) and search for solutions of the form $\varphi = \varphi_0\exp[i(\vec{k}\cdot\vec{r}+\omega t)]$, $\vec{V}_s = \vec{V}_{s0}\exp[i(\vec{k}\cdot\vec{r}+\omega t)]$. It can be shown that such solutions exist when

$$\omega k^2 = i\alpha[\Omega(k^2+k_z^2)+\nu k_z^2k^2] - \beta u k_z k^2 \pm \{(1-\beta)^2k_z^2k^2(2\Omega+\nu k_z^2)(2\Omega+\nu k^2) - \alpha^2\Omega^2(k^2-k_z^2)^2 - \alpha^2u^2k_z^2k^4 + 2i\alpha(1-\beta)uk_zk^2[\Omega(k^2+k_z^2)+\nu k_z^2k^2]\}^{1/2}, \tag{8}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. In the absence of mutual friction, this reduces to

$$\omega = \pm(k_z/k)[(2\Omega+\nu k_z^2)(2\Omega+\nu k^2)]^{1/2}. \tag{9}$$

The state of marginal stability is determined by the condition

$$\mathrm{Im}(\omega) = 0, \tag{10}$$

which determines the critical velocity for a mode of wave vector $\vec{k}$,

$$u_c = k^{-1}[(2\Omega+\nu k_z^2)(2\Omega+\nu k^2)]^{1/2}. \tag{11}$$

When the axial flow is at the critical value, the frequency is given by Eq. (9) so that the condition for instability can be started as follows: A mode of wave vector $\vec{k}$ is marginally stable when the projection of the normal-fluid velocity onto that wave vector is equal to the phase velocity of that mode.

The critical velocity for modes with wave vectors off-axis, that is oriented at an angle θ with respect to the rotational axis, decreases with increasing θ, although it is never less than $(2\Omega\nu)^{1/2}$. This comes about because the phase velocity decreases more rapidly with angle than $\cos\theta$. The effect of orienting the normal-fluid flow at some angle with respect to the rotational axis is to impose a uniform transverse translation of the array and to increase the critical velocity for the instability, but the component of the critical velocity along the rotation axis is unchanged. Normal flow transverse to the vortex array does not contribute to the instability.

In the experiments reported by Cheng, Cromar, and Donnelly,[4] a thermal counterflow was impressed along the axis of rotation in rotating helium. The attenuation of a transverse negative-ion beam—due to trapping of the ions on vortex lines—was found to decrease significantly as a result of the counterflow. We should like to explain these results in terms of the vortex-array instability. In a thermal-counterflow experiment (assuming Poiseuille flow for the normal fluid[10]), we can estimate the heat current necessary to begin disrupting the vortex array by

$$\dot{q}_c \sim \tfrac{1}{2}\rho_s ST(2\Omega\nu)^{1/2}, \qquad (12)$$

where ρ_s is the superfluid density,[11] S is the specific entropy, and T is the temperature. The points in Fig. 1 are the measured heat currents at which 20% of the recoverable ion beam was restored and the solid line is a plot of Eq. (12), where Ω is taken as 2.5 rad/sec. There is good qualitative and fair quantitative agreement between the theoretical and experimental results.

It should be pointed out that the instability begins when the maximum normal-fluid velocity reaches its critical value, where $\dot{q}$ is a measure of the average counterflow velocity. In particular, any nonuniformity in the heat current would tend to decrease the measured $\dot{q}_c$. We have only considered the normal modes of an unbounded system. Since the relevant wavelengths are much less than typical cell dimensions, boundary effects are likely to be small.

It is possible to use Eq. (8) to estimate the growth rate of the instability. Taking $\alpha \ll 1$ and assuming $\beta \sim 0$, one obtains, for a normal-fluid velocity equal to twice its critical value, $\mathrm{Im}(\omega) \approx -4\Omega\alpha$. The amplification of a wave propagat-

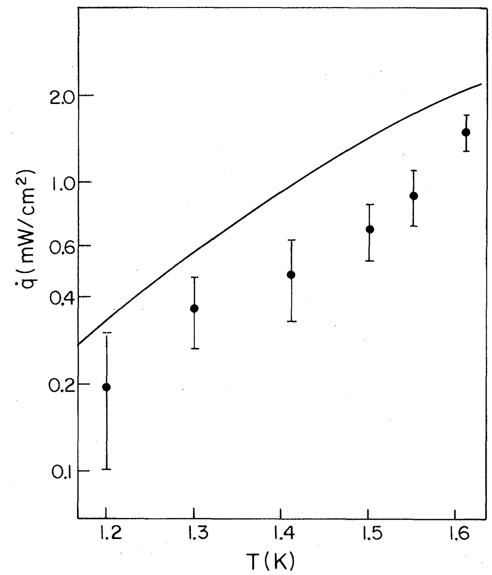

FIG. 1. Critical heat current for the onset of instability as a function of temperature. The points are the heat currents necessary to restore 20% of the recoverable ion currents, taken from smoothed data of Cheng, Cromar, and Donnelly (Ref. 4), and the solid line is a plot of the theoretical critical heat current from Eq. (12). The error bars on the experimental points are estimates of the scatter in the data.

ing in the z direction proceeds as $\exp(z/l)$, where

$$l \approx -\mathrm{Re}(\omega)/\mathrm{Im}(\omega)k_c \approx \alpha^{-1}(\nu/2\Omega)^{1/2}. \qquad (13)$$

At a temperature of 1.4 K, this leads to a characteristic length of ~0.27 cm so that in 5 cm, an amplification by a factor of more than 10^8 occurs.

The thermal fluctuations of a single vortex line may be estimated by allowing an energy of $K_B T$ in the critical oscillation mode:

$$L_0 \epsilon k_c^2 \delta^2/2 \sim K_B T, \qquad (14)$$

where L_0 is the length of the undeformed line, ϵ is the energy per unit length, and δ is the amplitude of the helical deformation of the line with wave vector k_c. This leads to an amplitude ~30 Å at a temperature of 1 K. Mechanical vibrations at 4Ω would probably be even more important for providing the initial deformation. In any event it is clear that for normal-fluid velocities close to the critical velocity, deformations of the vortex array on a scale as large as centimeters are certainly possible. Long before large-scale deformations occurred, a vortex tangle would probably develop which could release the trapped ions in a relatively short mean time.

We would expect an increase of second-sound attenuation associated with the vortex-line instability. Cheng, Cromar, and Donnelly[4] report that they saw no large effect on second sound in heat currents large enough to affect ion-beam attenuation. That part of their experiment was designed to determine whether or not a vortex array was still present and their experiment was not sensitive to an increased attenuation.[12]

The instability discussed in this paper may contribute to the anomalous lifetimes for negative ions on vortex lines at temperatures below 1 K observed by Douglass[13] as well as to the waving of vortex lines observed by Williams and Packard,[5] where the counterflow arises from stray heat currents.

In summary, we have shown that Hall's equations for a rotating superfluid, as well as simple physical arguments, imply that the vortex array becomes unstable for axial normal-fluid velocities greater than some critical value. This instability may have been observed in several recent experiments.

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