Transition to Turbulence of a Statically Stressed Fluid*

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The approach to and development of turbulence of low–Prandtl-number fluids in the Rayleigh-Bénard geometry is studied with truncated Boussinesq equations. Semiquantitative agreement is found with experimental results on helium for the turbulent threshold, and for the intensity and frequency spectrum of fluctuations at slightly higher Rayleigh numbers. It is argued that this transition conforms with abstract mathematical proposals of Ruelle and Takens, but that other transitions violate their picture.

The purpose of this Letter is to discuss the transition to nonperiodic fluctuations in a model of an ideal thermal-convection system, and to report on numerical studies of the fluctuations. These computations are in reasonable agreement with the experimentally observed fluctuations in a real convection system.[1] The calculated transition seems to agree in character with the qualitative picture of the transition to turbulence suggested by Ruelle and Takens.[2]

Thermal convection occurs in classical fluids when the component of an externally imposed temperature gradient along the direction of gravity exceeds a certain magnitude. This threshold gradient is determined by the condition that the buoyancy force to which it gives rise is sufficient to overcome the dissipative forces in the fluid.

In an infinite fluid layer of uniform cross section, the equations for the velocity and temperature fields contain two dimensionless parameters. These parameters are the Rayleigh number, R, and the Prandtl number, σ:

$$R \equiv g\epsilon H^3 \Delta T/\kappa\nu, \tag{1}$$

$$\sigma \equiv \nu/\kappa. \tag{2}$$

In Eqs. (1) and (2), g is the gravitational constant, ϵ is the coefficient of thermal expansion, H is the thickness of the fluid layer, ΔT is the temperature difference across the layer, κ is the thermal diffusivity, and ν is the kinematic viscosity.

It is well known from linear stability analysis on the conduction profile of a fluid layer with free surfaces[3] that when the Rayleigh number R exceeds a critical value, R_c, convective motion in the form of rolls develops. This critical Rayleigh number is independent of Prandtl number, and is equal to 657. The analogous calculation for a fluid subjected to rigid boundary conditions yields $R_c = 1707$.[4]

For a range of Rayleigh numbers above R_c, a steady-state time-independent convective motion develops. However, at sufficiently high Rayleigh numbers, this steady-state motion becomes unstable to time-dependent motion. This transition has been studied experimentally by Ahlers,[1] Krishnamurti,[5] and Willis and Deardorff.[6] Willis and Deardorff discovered that the nature of the transition to time dependence and its location in Rayleigh number depended on the Prandtl number. In the regime of low Prandtl numbers, e.g., for air for which $\sigma = 0.71$, Willis and Deardorff found a transition to wavy convection rolls. This instability and the succeeding ones leading to nonperiodic motion are the ones to be discussed in this Letter.

The linear stability analysis for the wavy instability was performed by Busse,[7] who studied the stability in the limit of vanishing Prandtl number. He showed that in this limit the system was extremely unstable; growing waves appeared on the rolls just beyond the Rayleigh number at which the rolls first formed. Busse also compared the linear stability analysis predicted by a particular eight–Fourier-component truncation of the system with the exact results. He found that the worst error produced by the truncation was about 20% of the exact result.

We have extended Busse's linear stability analysis to finite Prandtl numbers by using the eight-mode system which he proposed, and we have determined the threshold for the waves analytically by making an expansion in the wave number of the wave along the roll. The wave number can be chosen arbitrarily by imposing periodic boundary conditions along the length of the roll. In particular, we find for the threshold

$$r_T = 1 + 4\sigma^2(\sigma+1)/(13\sigma+21); \tag{3}$$

$$r = R/R_c, \quad r_T = R_T/R_c. \tag{4}$$

For values of the Rayleigh number larger than that given in Eq. (3), the linear theory predicts exponential growth of the wave. When the nonlinear corrections to the linear theory are included, a Landau expansion of the following form is obtained:

$$dA^2/dt = \alpha A^2 + \beta A^4 + \dots, \tag{5}$$

$$\alpha = \frac{2c_0 k^2}{\sigma}\left[\frac{13}{16} + \frac{1}{2(\sigma+1)}\right]\Delta c, \tag{6}$$

$$\beta = -k^2(5\sigma+13)/4\sigma(\sigma+1). \tag{7}$$

In Eqs. (5)–(7), all variables are dimensionless. The variable A is the amplitude of the wave, c_0 is the speed of the roll at r_T, Δc is the difference between the speed of the roll at r and its value at r_T, and k is the wave number of the wave along the roll.

For small amplitudes, all but the first two terms in the Landau expansion in Eq. (5) can be neglected. Since the coefficient of the A^4 term is always negative, the nonlinear corrections to the linear theory will saturate the growth of the wave, leading to a periodic motion with amplitude $A_s = (-\alpha/\beta)^{1/2}$.

Thus, a stable limit cycle is obtained with an amplitude that grows monotonically from zero as the Rayleigh number is increased above threshold. This behavior is consistent with the Hopf bifurcation theorem.[2] The Hopf bifurcation theorem states that, if a pair of complex-conjugate eigenvalues cross the imaginary axis, there will be a one-parameter family of limit-cycle solutions in the neighborhood of the point of neutral stability. This one-parameter family of limit cycles can lie on either the upper or the lower side of the point of neutral stability in stress. If the limit cycles lie on the lower side of the point of neutral stability ("inverted bifurcation"), they are unstable and lead to finite-amplitude instabilities and hysteresis phenomena. The three-mode model of time dependence in high–Prandtl-number convection which was studied by Lorenz[8] is an example of inverted bifurcation. In this model, there is an immediate transition to a complicated, nonperiodic motion. Even though the Lorenz model probably has little to do with the actual transition in high–Prandtl-number convection, it seems that this same feature of immediate transition to turbulence is shared by some other known examples of inverted bifurcation like pipe and channel flow.

The other possibility allowed by the Hopf bifurcation theorem is one in which the limit cycles lie above the point of neutral stability in stress ("normal bifurcation"). The wavy instability encountered in low–Prandtl-number convection is an example of normal bifurcation. One can ask the question of how nonperiodicity evolves in systems which exhibit normal bifurcation. A qualitative answer to this question has been proposed by Ruelle and Takens.[2] The idea is that, as the stress on a fluid system is increased, a succession of instabilities is encountered in which the degrees of freedom associated with harmonics of the basic wave numbers increase in size and complicate the time dependence of the system. Ruelle and Takens predicted that, after the second instability, the motion should generically still be periodic. However, they proposed that, after four instabilities, "strange attractors" should generically appear. Thus, Ruelle and Takens predict steady-state nonperiodic motion after four instabilities. The situation after three instabilities is nebulous.

The Ruelle-Takens picture of the transition to turbulence can be tested by calculating the time dependence of a model which contains the fundamental wave along the roll and its first three spatial harmonics. A consistent truncation of the Fourier spectrum includes modes of the velocity and temperature fields which have wave numbers that obey the following conditions:

$$|l| \leq 1; \quad |m| \leq 4; \quad |n| \leq 2; \qquad |l| + |n| \leq 2 \text{ and even.} \tag{8}$$

In Eq. (8), l is the wave number in the direction perpendicular to the roll axis and in the layer, measured in terms of the basic roll wave number. In the actual calculations, the basic roll wave number was chosen to be the most critical wave number which follows from the Rayleigh stability analysis. The variable m is the wave number along the roll axis measured in terms of the fundamental wave number in this direction. This fundamental wavelength was chosen to be roughly the circumference of the cylinder used in Ahlers's experiments. Finally, n is the dimensionless wave number in the vertical direction.

The truncation summarized in Eq. (8) leads to a model system consisting of 39 Fourier components (25 velocity and 14 temperature amplitudes). The reason that four harmonics are allowed along the roll axis and not in the other directions is that the fundamental wave number in this direction was chosen to be much smaller than the fundamental wave numbers in the other two directions. Thus, the damping due to the variations along the axis is small.

When the system of 39 first-order nonlinear ordinary differential equations in time is integrated on a computer, a succession of instabilities is found in which the higher-lying modes become large; the motion is strictly periodic until the fourth set of modes becomes large. There are two such instabilities. The first occurs at $r = 1.45$. At this Rayleigh number, the system is still in the "mean-field" regime, i.e., the rolls still have a preferred direction of rotation, and the fractional variation in heat flux is about 0.001.

The second transition to nonperiodicity occurs at $r = 1.6$. (The mean fields disappear at $r = 1.5$.) The fractional variation in heat flux is about 0.015 and is strongly nonperiodic at $r = 1.6$. The fractional variation in heat flux at $r = 1.5$ and 1.55 is about 0.0001 and is strictly periodic.

To test the Ruelle-Takens idea that four degrees of freedom are needed to obtain generic nonperiodicity, the modes having $m = 4$ may be removed from the model system. When this is done, the system reverts to exact periodicity at $r = 1.6$ and remains periodic at $r = 2$ and 20.

The nonperiodic fluctuations which develop at $r = 1.6$ in the computer calculations are in reasonably close agreement with the experimentally observed spectrum for liquid helium ($\sigma = 0.86$). Ahlers[1] found a transition to nonperiodic fluctuations in the heat flux at $r = 2.18$. The fluctuations were roughly 1% of the total heat flux. The worst discrepancy is that Ahlers's fluctuations were 2 or 3 times faster than the calculated fluctuations. Ahlers did not observe a periodic regime in the heat flux below $r = 2.18$. However, the existence of such a regime seems to be indicated in the experiments of Willis and Deardorff[6] on air ($\sigma = 0.71$). Rossby's experiments[9] on mercury ($\sigma = 0.025$), in which turbulence develops almost immediately at $r = 1$, also exhibit the Prandtl-number dependence of the model discussed here. However, Rossby's results seem to disagree with those of Krishnamurti,[10] who found that time dependence in mercury did not develop until $r = 1.4$.

Since the fluctuations and oscillations of the velocity are much greater than those in the total heat flux, valuable information could be provided by performing experiments like those of Berge and Dubois[11] above R_c and in the neighborhood of R_T with low–Prandtl-number fluids. Experiments with varying aspect ratio would also be useful.

Theoretically, more numerical studies should be performed and their reliability scrutinized. In particular, the "continuum" of modes with wave numbers close to those considered above, and close to zero wave number, should be examined. When the system is large enough for many of these modes to be significant, spatial dephasing of the rolls will have to occur. The frequency of these modes can provide an additional source of randomness in time that fits a generalized Landau-Lifshitz picture. We think, however, that the semiquantitative agreement with experiment and the Ruelle-Takens picture (as contrasted with that of Landau and Lifshitz and Hopf) is not entirely a numerical artifact. Preliminary calculations indicate that the bandwidth of the continuum is only 0.05 of the roll wave number at the transition to turbulence in the calculations reported here.

After this work was completed we learned of the unpublished work of Gough, Speigel, and Toomre.[12] They also studied the properties of slightly turbulent fluids with truncated Boussinesq equations. Their results appear to be primarily suited to high–Prandtl-number fluids and therefore complementary to ours. In the high–Prandtl-number fluids, thermal boundary layers are more significant and plumes are observed; far greater resolution in the vertical direction than we employed is necessary to describe these effects.

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Anisotropy of Self-Diffusion in the Smectic-*A* and Smectic-*C* Phases*

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The anisotropy of the translational self-diffusion tensor in the smectic-*A* ($\langle D_\parallel\rangle/\langle D_\perp\rangle = 0.3$) and smectic-*C* phases of terephthal-*bis*-4-*n*-butylaniline has been determined by multipulse NMR. The experimental results are interpreted in terms of a pseudolattice model with anisotropic potential barriers which seems to provide a better description of the physical situation than the "two-dimensional-liquid" model.

In this Letter we report what we believe to be the first measurement of the anisotropy of the self-diffusion tensor of the liquid crystal molecules in the smectic-*A* and smectic-*C* phases. The system investigated was terephthal-*bis*-4-*n*-butylaniline (TBBA).

Smectic-*A* and smectic-*C* liquid crystals[1] are usually considered to be two-dimensional nematic liquids where the molecules are free to move within the smectic layers. Within these equidistant layers the preferred direction of the long axis of each molecule (i.e., the molecular director) is parallel to the plane normals for smectic-*A* systems, whereas it is tilted with respect to the planes of the layers for smectic-*C* systems. In both smectic-*A* and smectic-*C* liquid crystals one would thus expect self-diffusion within the layers to be much larger than perpendicular to the smectic layers.

Murphy *et al.*[2] observed a large anisotropy in the self-diffusion coefficient of the spherical impurity molecule tetramethylsilane dissolved in a smectic-*A* system, but no measurement of the complete self-diffusion tensor of the smectic molecules themselves has been performed so far. Since only such a measurement can provide a quantitative test of the validity of the two-dimensional "free-flow" model within the smectic layers, it seemed worthwhile to use the newly developed multipulse line-narrowing proton-spin-echo technique for this purpose. The simple classical NMR technique for self-diffusion measurements is simply not applicable[3] in liquid crystalline systems because of a too short spin-spin relaxation time T_2.

The pulse sequence used which is similar to the one described earlier[3] is shown in Fig. 1. It consists of a (i) Waugh-type multiple 90° rf pulse sequence removing dipolar interactions: $-P_y-(t-P_x-2t-P_x-t-P_y-2t-P_y)_n$; (ii) a pulsed, linear, magnetic-field-gradient sequence placed between the rf pulses at such intervals that its effect is not averaged out by (i); (iii) a slow, refocusing, Carr-Purcell train of 180° rf pulses.

The echo maxima are given by the same expression as in Ref. 3. The components of the self-diffusion tensor $\underline{D}$ were determined by observing the dependence[3] of the spin-echo maxima at 87 msec on the strength and orientation of the field gradient $\vec{G} = \text{grad } H_Z$. The width of the gradient pulses varied between 2 and 8 μsec and the interval t between the 90° rf pulses at 60 MHz

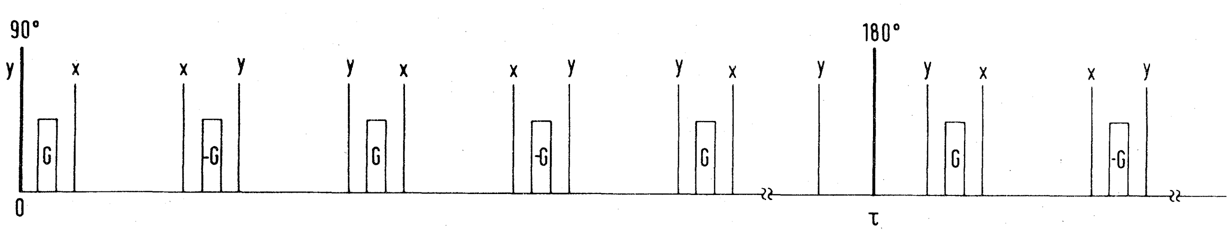

FIG. 1. Pulse sequence used for the determination of the translational self-diffusion tensor in systems with short spin-spin relaxation times T_2.